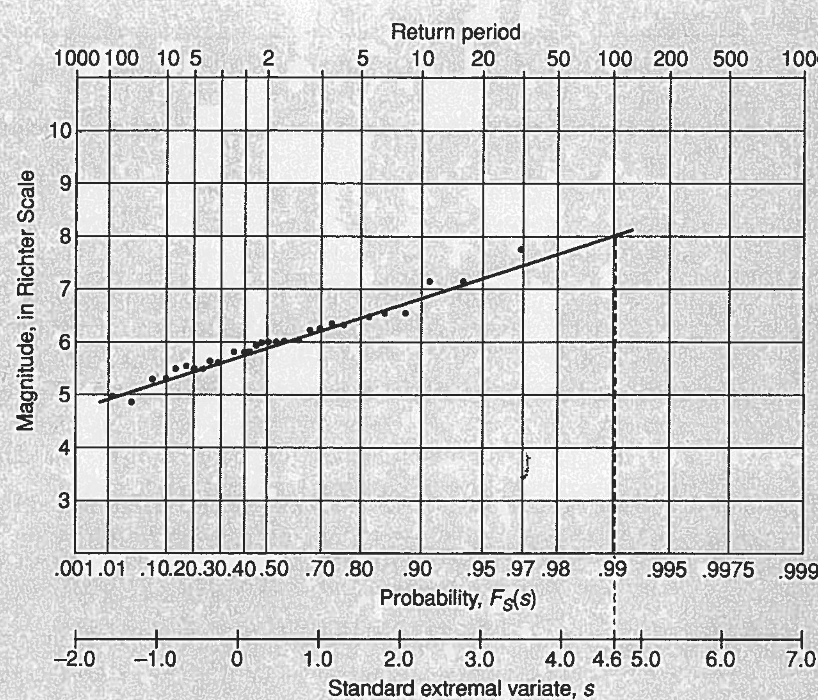

Figure E7.5 Annual largest earthquakes plotted on Gumbel probability paper. ◀

▶ 7.3 TESTING GOODNESS-OF-FIT OF DISTRIBUTION MODELS

When a particular probability distribution has been specified to model a random phenomenon, determined perhaps on the basis of available data plotted on a given probability paper, or through visual inspection of the shape of the histogram, the validity of the specified or assumed distribution model may be verified or disproved statistically by *goodness-of-fit* tests. Three such tests for distribution are available and used widely—the *chi-square, the Kolmogorov–Smirnov* (or K–S), and the *Anderson–Darling* (or A–D) methods; one or the other of these methods may be used to test the validity of a specified or assumed distribution model. When two (or more) distributions appear to be plausible models, the same test may be used also to discriminate the relative superiority between (or among) the assumed distribution models. The A–D test is particularly useful when the tails of a distribution are of importance.

7.3.1 The Chi-Square Test for Goodness-of-Fit

Consider a sample of n observed values of a random variable and an assumed probability distribution for modeling the underlying population. The chi-square goodness-of-fit test compares the *observed frequencies* $n_1, n_2, \ldots, n_k$ of k values (or in k intervals) of the variate with the corresponding *theoretical frequencies* $e_1, e_2, \ldots, e_k$ calculated from the assumed theoretical distribution model. The basis for appraising the goodness of this comparison is the distribution of the quantity

$$\sum_{i=1}^{k} \frac{(n_i - e_i)^2}{e_i}$$

which approaches the chi-square (χ_f^2) distribution with $f = k - 1$ degrees-of-freedom (d.o.f.) as $n \to \infty$ (Hoel, 1962). However, if the parameters of the theoretical model are unknown and need to be estimated also from the available data, the d.o.f. f must be reduced by one for every unknown parameter that must be estimated.

On the basis of the above, if the assumed distribution yields

$$\sum_{i=1}^{k} \frac{(n_i - e_i)^2}{e_i} < c_{1-\alpha,f} \tag{7.1}$$

in which $c_{1-\alpha,f}$ is the critical value of the χ_f^2 distribution at the cumulative probability of $(1-\alpha)$, the assumed theoretical distribution is an acceptable model, at the *significance level* α. Otherwise, if Eq. 7.1 is not satisfied, the assumed distribution model is not substantiated by the observed data at the α significance level. Values of $c_{1-\alpha,f}$ are tabulated in Table A.4.

In applying the chi-square test for goodness-of-fit, it is generally necessary for satisfactory results to have (if possible) $k \geq 5$ and $e_i \geq 5$.

▶ EXAMPLE 7.6

Severe thunderstorms have been recorded at a given station over a period of 66 years. During this period, the frequencies of severe thunderstorms observed are as follows:

20 years with zero storm	6 years with three storms
23 years with one storm	2 years with four storms
15 years with two storms	

The histogram of the annual number of rainstorms recorded at the station is displayed in Fig. E7.6. From the data, we estimate the mean annual occurrence rate of severe rainstorms to be 1.197 per year. In Fig. E7.6, we also display the Poisson distribution with an annual occurrence rate of $\nu = 1.197$, which appears to fit the observed histogram well.

Let us now apply the chi-square test to determine whether the Poisson distribution is a suitable model, at the 5% significance level. In this case, since four storms in a year were observed only twice, these data are combined with the data for three storms per year; thus, we have $k = 4$. Table E7.6 shows the calculations needed for the chi-square test.

Since the parameter ν is estimated from the observed data, the d.o.f. for the χ_f^2 distribution is $f = 4 - 2 = 2$. From Table A.4, we obtain at $(1-\alpha) = 0.95$, $c_{0.95,2} = 5.99$. And from Table E7.6,

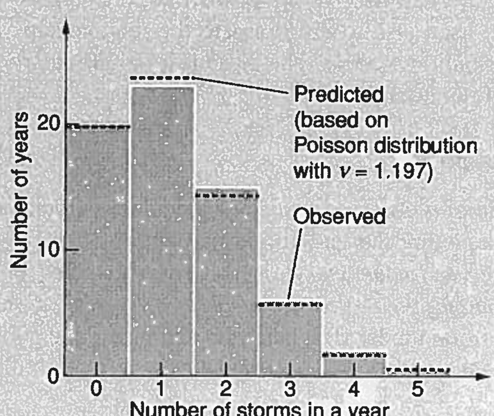

Figure E7.6 Histogram and the Poisson model for storm occurrences.

TABLE E7.6 Chi-Square Test of Poisson Model for Rainstorm Occurrences

No. of Storms per Year	Observed Frequencies, n_i	Theoretical Frequencies, e_i	$(n_i - e_i)^2$	$\frac{(n_i - e_i)^2}{e_i}$
0	20	19.94	0.0036	0.0002
1	23	23.87	0.7569	0.0317
2	15	14.29	0.5041	0.0353
≥3	8	7.90	0.0100	0.0013
$\sum$	66	66.00		0.0685

we have

$$\sum (n_i - e_i)^2/e_i = 0.068 < 5.99$$

Therefore, the Poisson distribution is suitable for modeling the annual occurrences of rainstorms at the station, at the 5% significance level. ◀

▶ EXAMPLE 7.7

Consider the histogram of the crushing strength of concrete cubes shown in Fig. E7.7. Also shown in the same figure are the normal and lognormal PDFs with the same mean and standard deviation as estimated from the observed data set. Visually, it appears that the two theoretical distributions are equally valid to model the crushing strength of concrete.

In this case, the chi-square test can be used to discriminate the relative goodness of fit between the two candidate distributions. For this purpose, eight intervals of the crushing strength are considered as indicated in Table E7.7.

The two parameters of both the normal and lognormal distributions were estimated from the sample data; consequently, the number of d.o.f. in both cases is $f = 8 - 3 = 5$. Therefore, at the significance level of $\alpha = 5\%$, we obtain from Table A.4, $c_{0.95,5} = 11.07$. Comparing this with the sum of the last two columns in Table E7.7, we see that both the normal and the lognormal are suitable for modeling the crushing strength of concrete; however, the lognormal model is superior to the normal model as indicated by the values of $\sum(n_i - e_i)^2/e_i$ for the two distributions, as shown in columns 5 and 6 of Table E7.7.

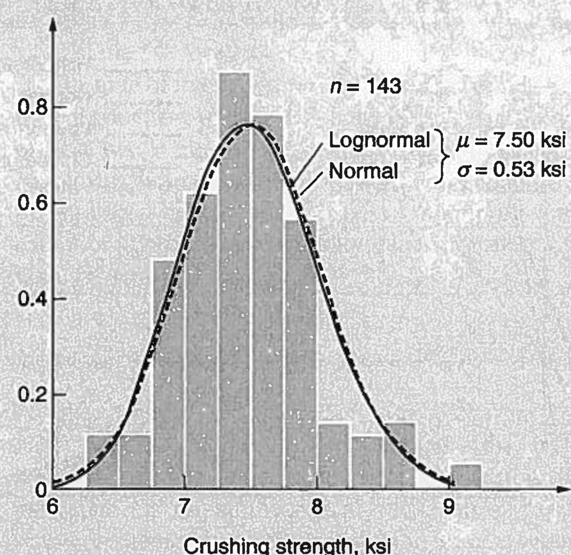

Figure E7.7 Histogram of crushing strength of concrete cubes. (Data from Cusens and Wettern, 1959.)

TABLE E7.7 Computations for Chi-Square Tests of Normal and Lognormal Distributions

Interval (ksi)	Observed Frequency, n_i	Theoretical Frequencies, e_i		$(n_i - e_i)^2/e_i$	
		Normal	Lognormal	Normal	Lognormal
< 6.75	9	11.1	9.9	0.40	0.09
6.75–7.00	17	13.2	14.0	1.09	0.92
7.00–7.25	22	21.1	22.1	0.04	0.00
7.25–7.50	31	26.1	26.9	0.92	0.62
7.50–7.75	28	26.1	25.6	0.14	0.23
7.75–8.00	20	21.0	19.8	0.05	0.00
8.00–8.50	9	20.2	19.4	6.22	5.57
> 8.50	7	4.2	5.3	1.87	0.54
$\sum$	143	143.0	143.0	10.73	7.97

◀

▶ **EXAMPLE 7.8**

From a sample of 320 observations, the histogram of residual stresses of wide flange steel beams is shown in Fig. E7.8. Superimposed on the same figure are the PDFs of three theoretical distribution models: the normal, lognormal, and the shifted (or 3-parameter) gamma distributions. From the data set, the first three moments were estimated to be, $\mu = 0.3561$, $\sigma = 0.1927$, and $\theta = 0.8230$ (skewness coefficient). Accordingly, the normal and lognormal distributions are assumed to have the estimated values for μ and σ, whereas the shifted gamma is assumed with the three estimated moments.

By visual inspection of Fig. E7.8, we can observe that the normal distribution is clearly not suitable. However, the lognormal and the shifted gamma distributions appear to fit the histogram well. In order to verify the relative validities of the three distribution models, we perform the chi-square test for goodness-of-fit with the calculations shown in Table E7.8.

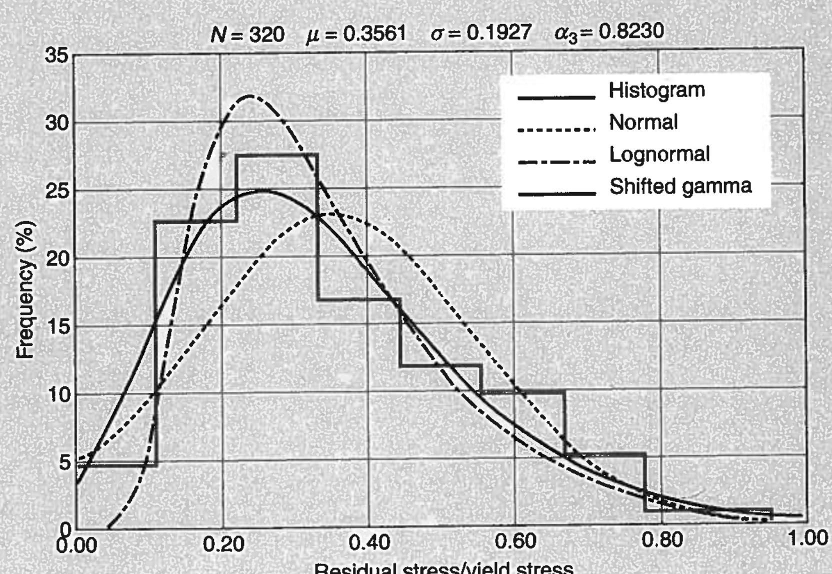

Figure E7.8 Chi-square tests to discriminate three distribution models for residual stresses.

7.3.

TABLE E7.8 Computations for Chi-Square Tests of Three Distributions

Interval	Observed	Theoretical Frequencies			$\Sigma(n_i-e_i)^2/e_i$		
Res. Str./Yield Str.	Frequency	Norm	Lognorm	Shifted Gamma	Norm	Lognorm	Shifted Gamma
< 0.111	15	32.6	6.5	23.2	9.48	10.90	2.88
0.111–0.222	72	45.4	73.2	61.0	15.60	0.02	1.97
0.222–0.333	88	67.0	95.9	78.1	6.57	0.66	1.25
0.333–0.444	54	71.6	66.0	66.5	4.33	2.17	2.35
0.444–0.555	38	55.4	37.1	44.2	5.44	0.02	0.87
0.555–0.666	31	31.0	19.5	24.9	0.00	6.75	1.51
0.666–0.777	16	12.5	10.1	12.4	0.96	3.40	1.03
0.777–0.888	3	3.7	5.3	5.7	0.12	0.99	1.27
> 0.888	3	0.9	6.3	4.0	4.81	1.75	0.25
$\sum$	320	320	320	320	47.3	26.7	13.4

From Table E7.8, we see that among the three distributions, the shifted gamma distribution gives the lowest value for $\sum(n_i - e_i)^2/e_i$. Also, at the significance level of 1% and a d.o.f. of $f = 9 - 4 = 5$, we obtain from Table A.4 the critical value of $c_{0.99,5} = 15.09$ for the normal and lognormal distributions, whereas for the shifted gamma distribution $f = 9 - 5 = 4$ and $c_{0.99,4} = 13.28$. Therefore, according to the chi-square test, only the shifted gamma distribution (among the three distributions) is approximately valid at the 1% significance level for modeling the probability distribution of residual stresses in wide-flange beams. ◄

It may be emphasized that because there is some arbitrariness in the choice of the significance level α, the chi-square goodness-of-fit test (as well as the Kolmogorov–Smirnov and the Anderson–Darling methods, described subsequently in Sects. 7.3.2 and 7.3.3) may not provide absolute information on the validity of a specific distribution. For example, it is conceivable that a distribution acceptable at one significance level may be unacceptable at another significance level; this can be illustrated with the shifted gamma distribution of Example 7.8, in which the distribution is valid at the 1% significance level but will not be valid at the 5% level.

In spite of this arbitrariness in the selection of the significance level, however, such statistical goodness-of-fit tests remain useful, especially for determining the relative goodness-of-fit of two or more theoretical distribution models, as illustrated in Examples 7.7 and 7.8. Moreover, these tests should be used only to help verify the validity of a theoretical model that has been selected on the basis of other prior considerations, such as through the application of appropriate probability papers, or even visual inspection of an appropriate PDF with the available histogram.

7.3.2 The Kolmogorov–Smirnov (K–S) Test for Goodness-of-Fit

Another widely used goodness-of-fit test is the *Kolmogorov–Smirnov* or K–S test. The basic premise of this test is to compare the experimental cumulative frequency with the CDF of an assumed theoretical distribution. If the maximum discrepancy between the experimental and theoretical frequencies is larger than normally expected for a given sample size, the theoretical distribution is not acceptable for modeling the underlying population; conversely, if the discrepancy is less than a critical value, the theoretical distribution is acceptable at the prescribed significance level α.

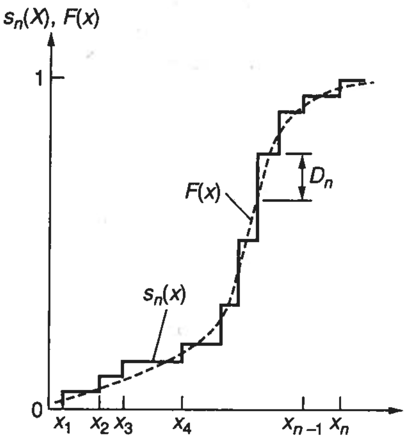

Figure 7.4 Empirical cumulative frequency versus theoretical CDF.

For a sample of size n, we rearrange the set of observed data in increasing order. From this ordered set of sample data, we develop a stepwise experimental cumulative frequency function as follows:

$$\begin{aligned} S_n(x) &= 0 \qquad x < x_1 \\ &= \frac{k}{n} \qquad x_k \leq x < x_{k+1} \\ &= 1 \qquad x \geq x_n \end{aligned} \tag{7.2}$$

where $x_1, x_2, \ldots, x_n$ are the observed values of the ordered set of data, and n is the sample size. Figure 7.4 shows a step-function plot of S_n and also an assumed theoretical CDF $F_X(x)$. In the K–S test, the maximum difference between $S_n(x)$ and $F_X(x)$ over the entire range of X is the measure of discrepancy between the assumed theoretical model and the observed data. Let this maximum difference be denoted by

$$D_n = \max_x |F_X(x) - S_n(x)| \tag{7.3}$$

Theoretically, D_n is a random variable. For a significance level α, the K–S test compares the observed maximum difference, D_n of Eq. 7.3, with the critical value D_n^α which is defined for significance level α by

$$P(D_n \leq D_n^\alpha) = 1 - \alpha \tag{7.4}$$

The critical values D_n^α at various significance levels α are tabulated in Table A.5 for various sample size n. If the observed D_n is less than the critical value D_n^α, the proposed theoretical distribution is acceptable at the specified significance level α; otherwise, the assumed theoretical distribution would be rejected.

The K–S test has some advantage over the chi-square test. With the K–S test, it is not necessary to divide the observed data into intervals; hence, the problem associated with small e_i and/or small number of intervals k in the chi-square test would not be an issue with the K–S test.

▶ EXAMPLE 7.9

The data for the fracture toughness of steel plates in Example 7.1 were plotted on a normal probability paper as shown in Fig. E7.1. The data appear to yield a linear trend producing a straight line corresponding to a normal distribution $N(77, 4.6)$. Let us now perform a K–S test to evaluate the

TABLE E7.9 Computations of S_n and $F_X(x)$ for Determining D

k	x	$S_n(x)$	$F_X(x) = \Phi\left(\frac{x-77}{4.6}\right)$	k	x	$S_n(x)$	$F_X(x) = \Phi\left(\frac{x-77}{4.6}\right)$
1	69.5	0.00	0.05	14	76.2	0.54	0.43
2	71.9	0.08	0.13	15	76.2	0.58	0.43
3	72.6	0.12	0.17	16	76.9	0.62	0.49
4	73.1	0.15	0.20	17	77.0	0.66	0.50
5	73.3	0.19	0.21	18	77.9	0.69	0.57
6	73.5	0.23	0.22	19	78.1	0.73	0.59
7	74.1	0.27	0.26	20	79.6	0.77	0.71
8	74.2	0.31	0.27	21	79.7	0.81	0.72
9	75.3	0.35	0.36	22	79.9	0.85	0.74
10	75.5	0.38	0.37	23	80.1	0.88	0.75
11	75.7	0.42	0.39	24	82.2	0.92	0.87
12	75.8	0.46	0.40	25	83.7	0.96	0.93
13	76.1	0.50	0.42	26	93.7	1.00	0.999

appropriateness of the proposed normal distribution model in light of the available data, at the 5% significance level.

With the tabulated data that have been rearranged in increasing order in Table E7.1, we illustrate the calculations of Eq. 7.2 for the empirical cumulative frequency and the corresponding theoretical CDF for the $N(77, 4.6)$ as shown in Table E7.9. The two cumulative frequencies are plotted as shown in Fig. E7.9. From Fig. E7.9, or Table E7.9, we can observe that the maximum discrepancy between the two cumulative frequencies is $D_n = 0.16$ and occurs at $x = K_{I_c} = 77$ ksi $\sqrt{\text{in}}$.

In this case, the sample size is $n = 26$; therefore, at the 5% significance level, we obtain the critical value of D_n^α from Table A.5 to be $D_{26}^{.05} = 0.265$. Since the maximum discrepancy $D_n = 0.16$ is less than 0.265, the normal distribution $N(77, 4.6)$ is verified as an acceptable model at the 5% significance level.

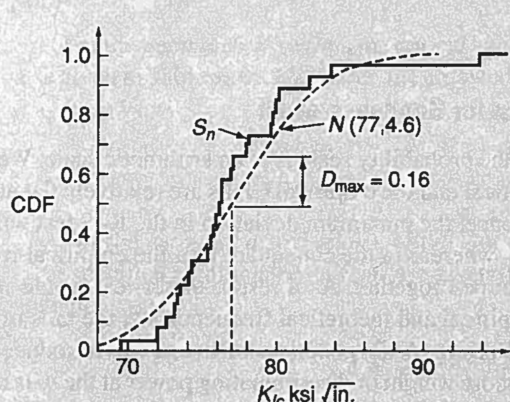

Figure E7.9 Cumulative frequencies for the K–S test of fracture toughness. ◀

▶ EXAMPLE 7.10

In Example 7.8, we tested the goodness-of-fit of three candidate distributions (the normal, lognormal, and the shifted gamma) for the residual stresses in wide flange steel beams using the chi-square test. With the same data, let us now use the K–S test to determine the goodness-of-fit of the same three distributions.

For the K–S test, we constructed Fig. E7.10 to display the empirical cumulative frequency function of the observed data and the respective CDFs of the normal, lognormal, and shifted gamma distributions. The respective maximum discrepancies for the three theoretical distributions, in accordance with Eq. 7.3, are found to be as follows:

For the normal:	$D_n = 0.1148$
For the lognormal:	$D_n = 0.0785$
For the shifted gamma:	$D_n = 0.0708$

With the sample size of $n = 320$ and at the significance level of 5%, we obtain from Table A.5 the critical value of $D_{320}^{.05} = 1.36/\sqrt{n} = 1.36/\sqrt{320} = 0.0760$. Comparing the above D_n values for the respective distributions with this critical value, we see that at the significance level of 5%, the shifted gamma distribution (with $D_n = 0.0708 < 0.0760$) is verified to be acceptable for modeling the residual stresses, whereas the normal and lognormal distributions must be rejected.

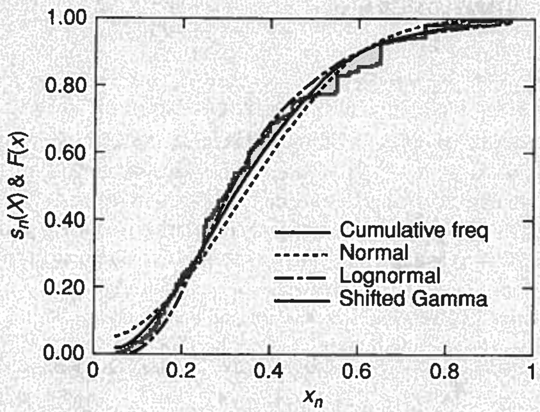

Figure E7.10 Empirical cumulative frequency and the CDFs of three distribution models ◀

7.3.3 The Anderson–Darling (A–D) Test for Goodness-of-Fit

In the K–S test, the probability scale is in an arithmetic scale. We may observe that both the proposed theoretical and the empirical CDFs are relatively flat at the tails of the probability distributions. Hence, the maximum deviation in the K–S test will seldom occur in the tails of a distribution, whereas in the chi-square test, the empirical frequencies at the tails must generally be grouped together. As a result, either test would not reveal any discrepancy between the empirical and theoretical frequencies at the tails of the proposed distribution. The Anderson–Darling (A–D) goodness-of-fit test was introduced by Anderson and Darling (1954) to place more weight or discriminating power at the tails of the distribution. This can be important when the tails of a selected theoretical distribution are of practical significance. The procedure for applying the A–D method can be described with the following steps:

1. Arrange the observed data in increasing order: $x_1, x_2, \ldots, x_i, \ldots, x_n$, with x_n as the largest value.
2. Evaluate the CDF of the proposed distribution $F_X(x_i)$ at x_i, for $i = 1, 2, \ldots n$.
3. Calculate the Anderson–Darling (A–D) statistic

$$A^2 = -\sum_{i=1}^{n}[(2i-1)\{\ln F_X(x_i) + \ln[1 - F_X(x_{n+1-i})]\}/n] - n \qquad (7.5)$$

4. Compute the adjusted test statistic A^* to account for the effect of sample size n. This adjustment will depend on the selected form of the distribution; see Eqs. 7.7 through 7.10.
5. Select a significance level α and determine the corresponding critical value c_α for the appropriate distribution type. Values of c_α are given in Table A.6 (*a* through *d*) for four common distributions.
6. For a given distribution, compare A^* with the appropriate critical value c_α. If A^* is less than c_α, the proposed distribution is acceptable at the significance level α.

The test is valid for a sample size larger than 7; i.e, $n > 7$. Because the A–D statistic is expressed in terms of the logarithm of the probabilities, it would receive more contributions from the tails of a distribution. We emphasize that the critical value, c_α, in the A–D test for a given significance level α depends on the form of the proposed theoretical distribution. Also the adjusted A–D statistic, A^*, will depend on the sample size n. These are given and defined in Table A.6, specifically, as follows:

For the *normal distribution*, the critical value c_α is given by

$$c_\alpha = a_\alpha\left(1 + \frac{b_0}{n} + \frac{b_1}{n^2}\right) \tag{7.6}$$

in which values of a_α, b_0, and b_1 are given in Table A.6a for a prescribed significance level α, and the adjusted A–D statistic for a sample size n is

$$A^* = A^2\left(1.0 + \frac{0.75}{n} + \frac{2.25}{n^2}\right) \tag{7.7}$$

For the *exponential distribution*, the critical value of c_α at a specified significance level α is given in Table A.6b, and the corresponding adjusted A–D statistic, A^*, is given by

$$A^* = A^2\left(1.0 + \frac{0.6}{\sqrt{n}}\right) \tag{7.8}$$

For the *gamma distribution*, the critical value of c_α depends on the value of the parameter k as given in Table A.6c. Moreover, the adjusted A–D statistic also depends on the parameter k as follows:

$$\begin{aligned} &\text{For } k = 1, && A^* = A^2(1.0 + 0.6/n) \\ &\text{For } k \geq 2 && A^* = A^2 + (0.2 + 0.3/k)/n \end{aligned} \tag{7.9}$$

For the *extremal distributions*, of the Gumbel and Weibull types, the critical values of c_α at specified significance levels α are given in Table A.6d. In this case, the adjusted A–D statistic is given by

$$A^* = A^2\left(1.0 + \frac{0.2}{\sqrt{n}}\right) \tag{7.10}$$

► EXAMPLE 7.11

The steel toughness data of Example 7.1 was previously fitted with a normal distribution, and its goodness-of-fit was validated in Example 7.9 by the K–S Test. With the same data, we will now perform a validation of the normal distribution with the A–D Test at the 5% significance level. Table E7.11 summarizes the result of the calculations according to the procedure outlined earlier. The proposed normal model is N(76.99, 4.709) with the parameters, μ and σ, estimated from the sample of size 26. From the results in Table E7.11, we calculate the A–D statistic as

$$A^2 = -\frac{-699.476}{26} - 26 = 0.903$$

TABLE E7.11 Computations for the Anderson-Darling Test of the Normal Distribution

i	x_i	$F_X(x_i)$	$F_X(x_{27-i})$	$(2i\text{-}1)\{\ln F_X(x_i)+\ln[1-F_X(x_{27-i})]\}$
1	69.5	0.055787	0.999806	−11.4342
2	71.9	0.139745	0.922853	−13.5899
3	72.6	0.175460	0.865630	−18.7375
4	73.1	0.204226	0.745369	−20.6953
5	73.3	0.216478	0.731552	−25.6083
6	73.5	0.229144	0.717368	−30.1072
7	74.1	0.269526	0.710143	−33.1429
8	74.2	0.276587	0.592990	−32.7622
9	75.3	0.359648	0.576430	−31.9883
10	75.5	0.375650	0.500652	−31.7974
11	75.7	0.391869	0.492180	−33.9035
12	75.8	0.400052	0.433188	−34.1294
13	76.1	0.424850	0.433188	−35.5937
14	76.2	0.433188	0.424850	−37.5221
15	76.2	0.433188	0.400052	−39.0774
16	76.9	0.492180	0.391869	−37.3946
17	77.0	0.500652	0.375650	−38.3753
18	77.9	0.576430	0.359648	−34.8824
19	78.1	0.592990	0.276587	−31.3151
20	79.6	0.710143	0.269526	−25.5977
21	79.7	0.717368	0.229144	−24.2892
22	79.9	0.731552	0.216478	−23.9313
23	80.1	0.745369	0.204226	−23.5042
24	82.2	0.865630	0.175460	−15.8497
25	83.7	0.922853	0.139745	−11.3098
26	93.7	0.999806	0.055787	−2.9375
				$\sum = -699.476$

and the corresponding adjusted statistic becomes

$$A^* = 0.903\left(1.0 + \frac{0.75}{26} + \frac{2.25}{26^2}\right) = 0.932$$

From Table A.6a, we obtain at the significance level of $\alpha = 0.05$, the values for a_α, b_0, and b_1 as $a_\alpha = 0.7514$; $b_0 = -0.795$; and $b_1 = -0.890$, yielding the critical value of

$$c_{0.05} = 0.7514\left(1 + \frac{-0.795}{26} + \frac{-0.890}{26^2}\right) = 0.727$$

Since $A^* > 0.727$, the normal distribution is not acceptable at the 5% significance level. However, at the 1% significance level, the constants a_α, b_0, and b_1 from Table A.6a are 1.0348, −1.013, and −0.93 respectively. Hence, the critical value is $c_{0.01} = 0.994$. Therefore, at the 1% significance level, since $A^* < c_{0.01}$, the normal distribution would be acceptable. ◀

We might point out that the A–D test described above for the normal distribution is applicable also to the lognormal distribution. As the logarithm of a lognormal variate is normally distributed, we simply need to take the logarithms of the sample values of the variate in applying the same A–D test as for the normal distribution. That is, all the computations for a normal distribution would remain the same except that the sample values of the variate must be replaced by the respective logarithms of the variate; e.g., in Table E7.11, x_i must be replaced by the corresponding $\ln(x_i)$.

▶ **EXAMPLE 7.12**

As shown in Fig. E7.5, the annual largest earthquake magnitudes in California observed between 1932 and 1962 show a linear trend on the Gumbel probability paper. On this basis, the Gumbel distribution with the parameters $u = 5.7$ and $\alpha = 2.0$ is a viable model for the annual maximum earthquakes in California. We will now perform an A–D test for goodness-of-fit of this distribution at the 5% significance level.

The required calculations are summarized in Table E7.12.

Therefore, according to Eq. 7.5, the A–D statistic is

$$A^2 = -\frac{-976.86}{31} - 31 = 0.512$$

and the adjusted statistic according to Eq. 7.10 is

$$A^* = 0.512\left(1.0 + \frac{0.2}{\sqrt{31}}\right) = 0.530$$

TABLE E7.12 Computations for the Anderson–Darling Test of the Gumbel Distribution

i	x_i	$F_X(x_i)$	$F_X(x_{32-i})$	$(2i-1)\{\ln F_X(x_i) + \ln[1-F_X(x_{32-i})]\}$
1	4.9	0.00706	0.98185	−8.962
2	5.3	0.10801	0.94100	−15.167
3	5.3	0.10801	0.94100	−25.279
4	5.5	0.22496	0.81718	−22.338
5	5.5	0.22496	0.81718	−28.720
6	5.5	0.22496	0.78146	−33.138
7	5.5	0.22496	0.78146	−39.164
8	5.6	0.29482	0.73993	−38.523
9	5.6	0.29482	0.73993	−43.660
10	5.6	0.29482	0.69220	−45.594
11	5.8	0.44099	0.69220	−41.938
12	5.8	0.44099	0.57764	−38.654
13	5.8	0.44099	0.57764	−42.015
14	5.9	0.51154	0.57764	−41.370
15	6.0	0.57764	0.57764	−40.910
16	6.0	0.57764	0.57764	−43.732
17	6.0	0.57764	0.57764	−46.553
18	6.0	0.57764	0.51154	−44.286
19	6.0	0.57764	0.44099	−41.825
20	6.0	0.57764	0.44099	−44.086
21	6.2	0.69220	0.44099	−38.928
22	6.2	0.69220	0.29482	−30.839
23	6.3	0.73993	0.29482	−29.272
24	6.3	0.73993	0.29482	−30.573
25	6.4	0.78146	0.22496	−24.571
26	6.4	0.78146	0.22496	−25.573
27	6.5	0.81718	0.22496	−24.207
28	6.5	0.81718	0.22496	−25.121
29	7.1	0.94100	0.10801	−9.981
30	7.1	0.94100	0.10801	−10.331
31	7.7	0.98185	0.00706	−1.550
				$\sum = -976.86$

Table A.6d $\quad C_\alpha = 0.757$

$A^* < C_\alpha \Rightarrow$ Gumbel is acceptable

TABLE A.5 Critical Values of D_n^α at Significance Level α in the K-S Test

d.o.f. $= n$	$\alpha = 0.20$	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.01$
5	0.45	0.51	0.56	0.67
10	0.32	0.37	0.41	0.49
15	0.27	0.30	0.34	0.40
20	0.23	0.26	0.29	0.36
25	0.21	0.24	0.27	0.32
30	0.19	0.22	0.24	0.29
35	0.18	0.20	0.23	0.27
40	0.17	0.19	0.21	0.25
45	0.16	0.18	0.20	0.24
50	0.15	0.17	0.19	0.23
>50	$1.07/\sqrt{n}$	$1.22/\sqrt{n}$	$1.36/\sqrt{n}$	$1.63/\sqrt{n}$

TABLE A.6 Critical Values of the Anderson-Darling Goodness-of-Fit Test

Table A.6a Critical Values of c_α at Significance Level α of the A-D Test for the Normal Distribution (μ and σ Estimated from Sample Size n)

Significance Level α	a_α	b_0	b_1
0.2	0.5091	−0.756	−0.39
0.1	0.6305	−0.75	−0.8
0.05	0.7514	−0.795	−0.89
0.025	0.8728	−0.881	−0.94
0.01	1.0348	−1.013	−0.93
0.005	1.1578	−1.063	−1.34

Excerpted from D'Agostino and Stephens, 1986 (see References in Chapter 7).

Table A.6b Critical Values of c_α at Significance Level α of the A-D Test for the Exponential Distribution (Parameter λ Estimated from) Sample Size n)

Significance Level α	c_α
0.25	0.736
0.20	0.816
0.15	0.916
0.10	0.162
0.05	1.321
0.025	1.591
0.01	1.959
0.005	2.244
0.0025	2.534

Excerpted from Pearson and Hartley, 1972, and D'Agostino and Stephens, 1986 (see References in Chapter 7).

Table A.6c Critical Values of c_α at Significance Level α of the A-D Test for the Gamma Distribution (Parameters Estimated from Sample Data)

	Significance Level α					
k	0.25	0.10	0.05	0.025	0.01	0.005
1	0.486	0.657	0.786	0.917	1.092	1.227
2	0.477	0.643	0.768	0.894	1.062	1.190
3	0.475	0.639	0.762	0.886	1.052	1.178
4	0.473	0.637	0.759	0.883	1.048	1.173
5	0.472	0.635	0.758	0.881	1.045	1.170
6	0.472	0.635	0.757	0.880	1.043	1.168
8	0.471	0.634	0.755	0.878	1.041	1.165
10	0.471	0.633	0.754	0.877	1.040	1.164
12	0.471	0.633	0.754	0.876	1.038	1.162
15	0.470	0.632	0.754	0.876	1.038	1.162
20	0.470	0.632	0.753	0.875	1.037	1.161
∞	0.470	0.631	0.752	0.873	1.035	1.159

Excerpted from Lockhart and Stephens, 1985 (see References in Chapter 7).

Table A.6d Critical Values of c_α at Significance Level α of the A-D Test for the Gumbel and Weibull Distribution (Parameters Estimated from Sample Size n)

Significance Level α	c_α
0.25	0.474
0.10	0.637
0.05	0.757
0.025	0.877
0.01	1.038

Excerpted from Stephens, 1977 (see References in Chapter 7).