CHAPTER 2

Generalized Univariate Time-Series Analysis in Hydrology

2.1 INTRODUCTION AND NOTATION

The records of the measure of some hydrologic process in time constitute a time series. One of the aims of time-series analysis is to identify the mechanism which gives origin to the different values in time. Identification is never perfect, however, because of the limited duration of the phenomenon. Therefore in constructing a model for a specific purpose, the time-series analyst attempts to preserve the properties relevant to the problems with which he or she is dealing. In order to do this, he must identify the parameters leading to such properties and then obtain the best possible estimators of these parameters, based on the history of the phenomenon.

Models of hydrologic time series, such as streamflow and rainfall at one point, have played a major role in the water-resources literature over the past 20 years. These models are constructed to facilitate the analysis and planning of various water uses through Monte Carlo simulation or to help in the real-time operation of water works by forecasting hydrologic events.

The synthetic generation of hydrologic series (Monte Carlo simulation) was popularized in the early 1960s by the work of the Harvard Water Program

(Maass et al., 1962). Synthetic streamflows are used in problems of reservoir design and of operation of river-basin water-resource systems. The aim of synthetic hydrologic simulation is to produce a set of equally likely traces (as long as needed) that are statistically indistinguishable from the historical data. These traces show many possible hydrologic conditions that do not explicitly appear in the historical record. Different designs and operational schemes can then be tested under the many different conditions contained in the synthetic record. The study of the Delaware River by Hufschmidt and Fiering (1966) has become the classic example of operational, synthetic, or stochastic hydrology. In 1914, Hazen suggested combining data from several recording stations in order to augment available information and improve the design of municipal water-supply reservoirs. Sudler (1927) used cards to generate independent sequences of streamflow to use in streamflow-regulating reservoirs. Hurst (1951) interpreted annual streamflows as a random sequence and studied the implication of that stochasticity in storage needs to control the Nile River in Egypt.

The use of stochastic time-series models for hydrologic forecasting has come about more recently. A good review of the literature can be found in the papers presented at the Workshop on Recent Developments in Real Time Forecasting/Control of Water Resource Systems (Wood, 1980). Of unique historical importance is the work of Jamieson et al. (1972a, 1972b, 1976); this is the first clear attempt to control a multipurpose river reservoir system using stochastic models to forecast hydrologic events.

One of the most complete popular textbooks on time-series analysis is that by G. E. Box and G. M. Jenkins (1976), where the general class of autoregressive integrated moving average models (ARIMA) is studied in detail. This chapter will be based mostly on their work. We will not attempt to cover every aspect of the theory; more than one book exists on the topic. We wish to introduce the reader to the concepts of generalized time-series–analysis theory and to provide an operational knowledge of the available tools. This should be enough for most practicing hydrologists and will be a good start for those interested in pursuing the topic further. Examples of the applications of ARIMA models in hydrology will be given.

Definitions

Understanding generalized time-series analysis (in the time domain) requires knowing some basic definitions.

A linear filter is an operator (linear) that converts a sequence of uncorrelated (usually normal) random variables a_t to a sequence of correlated variables Z_t. This is illustrated in Fig. 2.1. The linear operator representing the filter will be called (following Box and Jenkins' notation) $\psi(B)$. The sequence of uncorrelated discrete random variables a_t will be referred to as white noise.

The linear operator $\psi(B)$ is a polynomial in the backward shift operator B. If B operates on the random variable Z_t it yields the random variable Z_{t-1},

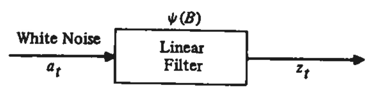

Figure 2.1 A linear filter acting on uncorrelated noise to create a time series (from Box and Jenkins, 1976).

essentially the process delayed by one discrete time unit. Mathematically,

$$BZ_t = Z_{t-1}.$$

Powers of B are logically defined,

$$B^2 Z_t = BBZ_t = BZ_{t-1} = Z_{t-2}.$$

In general,

$$B^m Z_t = Z_{t-m}. \quad (2.1)$$

A polynomial in B operates in a similar manner; for example,

$$(a_0 + a_1 B + a_2 B^2) Z_t = a_0 Z_t + a_1 Z_{t-1} + a_2 Z_{t-2}. \quad (2.2)$$

A forward shift operator F shifts the random process one time unit into the future,

$$FZ_t = Z_{t+1} \quad (2.3)$$

or

$$F^m Z_t = Z_{t+m}. \quad (2.4)$$

Premultiplying Eq. (2.3) by B,

$$BFZ_t = BZ_{t+1} = Z_t,$$

which implies $F = B^{-1}$.

The backwards difference operator ∇ is defined as

$$\begin{aligned} \nabla Z_t &= Z_t - Z_{t-1} \\ &= (1 - B) Z_t. \end{aligned} \quad (2.5)$$

Powers of ∇ are also well defined,

$$\begin{aligned} \nabla^2 Z_t &= \nabla \nabla Z_t = \nabla (Z_t - Z_{t-1}) \\ &= \nabla Z_t - \nabla Z_{t-1} \\ &= Z_t - Z_{t-1} - Z_{t-1} + Z_{t-2} \\ &= Z_t - 2Z_{t-1} + Z_{t-2} \\ &= (1 - B)^2 Z_t. \end{aligned} \quad ($$

The inverse of the backward difference operator ∇^{-1} is the back\ summation operator S. S is defined as

$$\begin{aligned} SZ_t &= \sum_{j=0}^{\infty} Z_{t-j} \\ &= (1 + B + B^2 + \cdots) Z_t. \end{aligned} \quad ($$

2.1.1 The General Stationary Linear Filter (Model)

Figure 2.1 represented the general linear model with operand $\psi(B)$. Ass from now on that the output Z_t is a stationary process that has t normalized by subtracting its constant mean from all its discrete values; Z_t zero-mean process. Therefore Z_t is related to the white noise a_t by a disc convolution.

$$\begin{aligned} Z_t &= a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots \\ &= a_t + \sum_{j=1}^{\infty} \psi_j a_{t-j} \\ &= \sum_{j=0}^{\infty} \psi_j a_{t-j} \\ &= \psi(B) a_t, \end{aligned} \quad (:$$

where $\psi_0 = 1$ and

$$\psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots.$$

The sequence of white noise a_t has zero mean and variance, σ_a^2; autocovariance function of a_t is, then,

$$\gamma_k = E[a_t a_{t+k}] = \begin{cases} \sigma_a^2 & k = 0 \\ 0 & k \neq 0 \end{cases}, \quad (:$$

which implies that the autocorrelation function ρ_k, takes a value of 1 at $k=0$ and is zero elsewhere.

Using Eq. (2.8) and the properties of a_t, the covariance function of Z_t becomes,

$$\gamma_k = E[Z_t Z_{t+k}] = E\left[\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\psi_j\psi_h a_{t-j}a_{t+k-h}\right] = \sigma_a^2\sum_{j=0}^{\infty}\psi_j\psi_{j+k}. \quad (2.10)$$

For $k=0$, the above equation yields the variance of the process

$$\gamma_0 = \sigma_z^2 = \sigma_a^2\sum_{j=0}^{\infty}\psi_j^2. \quad (2.11)$$

Unless the infinite summation of Eq. (2.11) converges, the variance of the generalized linear filter would be infinite. Finite variance and stationarity will require convergence of the polynomial $\psi(B)$ for $|B|\le 1$ (see Box and Jenkins, 1976). In the above statement, the operator B in Eq. (2.8) is interpreted as a dummy variable, which may take complex values. The stationarity condition then states that the infinite polynomial $\psi(B)$ must converge for all values of B within or on the unit circle, $|B|\le 1$.

The linear stationary filter given in Eq. (2.8) is a moving average of infinite past white-noise sequences. It is possible to express the same process as an infinite autoregressive sequence,

$$Z_t = \pi_1 Z_{t-1} + \pi_2 Z_{t-2} + \cdots + a_t = \sum_{j=1}^{\infty}\pi_j Z_{t-j} + a_t$$
$$\text{or,}\quad \pi(B)Z_t = a_t \quad (2.12)$$

where $\pi(B) = 1 - \pi_1 B - \pi_2 B^2 - \cdots$.

To illustrate the equivalence between the moving average and autoregressive model, assume the following model,

$$Z_t = (1-\theta B)a_t = \psi(B)a_t, \quad (2.13)$$

which is stationary for any value of θ.

By iteratively substituting for the a_t terms, Eq. (2.13) can be expressed as

$$a_t = Z_t + \theta Z_{t-1} + \theta^2 Z_{t-2} + \cdots + \theta^n a_{t-n}. \quad (2.14)$$

If $|\theta|>1$, Z_t depends on the past values with weights that increase v distance into the past. To avoid this situation, the invertibility condit $|\theta|<1$, is imposed. This allows the association of the present with the past more sensible manner. Such a condition is equivalent to stating that $(1-\theta B$ can be expressed as a convergent infinite geometric series, $(1+\theta B+\theta^2 B$ $\theta^3 B^3 + \cdots)$.

By analogy to Eq. (2.12) the autoregressive version of the model in (2.13) has coefficients given by

$$\pi_j = -\theta^j \quad \text{for } j=1,\ldots,\infty. \quad (2.$$

Generalizing, a model is invertible if the polynomial in B,

$$\pi(B) = \psi^{-1}(B), \quad (2.$$

converges for all $|B|\le 1$ (within or on the unit circle).

2.2 AUTOREGRESSIVE MODELS OF ORDER p, AR(p)

Autoregressive models of infinite order, such as that given in Eq. (2.12), are little use in practice. A finite-order model is more useful,

$$(1-\phi_1 B - \cdots - \phi_p B^p)Z_t = a_t$$
$$\phi(B)Z_t = a_t. \quad (2.$$

An example would be a lag-one model of the form

$$(1-\phi_1 B)Z_t = a_t. \quad (2.$$

Because the polynomial $\phi(B)$ is finite, Eq. 2.18 is unconditionally inve ible, a property held by all AR models of finite order p.

Stationarity can be investigated by reformulating the model in the form Eq. (2.8),

$$Z_t = (1-\phi_1 B)^{-1}a_t = \sum_{j=0}^{\infty}\phi_1^j a_{t-j} = (1+\phi_1 B+\phi_1^2 B^2+\cdots)a_t = \psi(B)a_t, \quad (2.1$$

which shows that a finite AR model is equivalent to an infinite moving avera We have seen that for the infinite-moving-average model to be stationa $\psi(B)$ must converge for $|B|\le 1$. For the AR(1) model given above, this impl that $|\phi_1|<1$ is required to ensure stationarity. Note that the root of $1-\phi_1 B =$ is $B=\phi_1^{-1}$. Since $|\phi_1|<1$, then the stationarity condition is that the root of t polynomial in B, $\phi(B)$, must lie outside the unit circle. The above conclusion is valid for autoregressive models of any order.

The autocovariance function of AR(p) is obtained by multiplying $Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t$ by Z_{t-k} and taking expected values:

$$E[Z_{t-k}Z_t] = E[Z_{t-k}(\phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t)]$$
$$= E[\phi_1 Z_{t-k}Z_{t-1} + \phi_2 Z_{t-k}Z_{t-2} + \cdots + \phi_p Z_{t-k}Z_{t-p} + Z_{t-k}a_t]. \tag{2.20}$$

Using the lack of correlation between Z_{t-k} and a_t and invoking stationarity, the autocovariance at lag k, γ_k, becomes

$$\gamma_k = \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2} + \cdots + \phi_p\gamma_{k-p}, \tag{2.21}$$

where $\gamma_{k-i} = E[Z_{t-k}Z_{t-q}]$.

Dividing by the stationary process variance, γ_0, yields

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} + \cdots + \phi_p\rho_{k-p}. \tag{2.22}$$

Therefore, the autocorrelation of an AR(p) obeys the equation

$$\phi(B)\rho_k = 0.$$

Writing Eq. (2.22) for $k = 1, 2, \ldots, p$ yields the Yule–Walker system of equations

$$\begin{aligned}
\rho_1 &= \phi_1 &+ \phi_2\rho_1 &+ \cdots + \phi_p\rho_{p-1}\\
\rho_2 &= \phi_1\rho_1 &+ \phi_2 &+ \cdots + \phi_p\rho_{p-2}\\
\vdots &\quad \vdots & & \quad\vdots\\
\rho_p &= \phi_1\rho_{p-1} &+ \phi_2\rho_{p-2} &+ \cdots + \phi_p
\end{aligned} \tag{2.23}$$

Note that in formulating the above system, the identities $\rho_0 = 1$ and $\rho_k = \rho_{-k}$ were used. The Yule–Walker equations can be used to estimate parameters of AR(p) by substituting sample correlations, r_k, in Eq. (2.23) and solving for the ϕs. Given that sample correlations are not very stable estimates, particularly for high lags, care must be taken in using this parameter-estimation procedure. Note that in matrix form

$$\mathbf{\Phi} = \mathbf{\Sigma}^{-1}\mathbf{R}, \tag{2.24}$$

where

$$\mathbf{\Phi} = \begin{bmatrix}\phi_1\\ \vdots\\ \phi_p\end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix}\rho_1\\ \vdots\\ \rho_p\end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix}1 & \rho_1 & \rho_2 & \cdots & \rho_{p-1}\\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{p-2}\\ \vdots & \vdots & \vdots & & \vdots\\ \rho_{p-1} & \rho_{p-2} & \rho_{p-3} & \cdots & 1\end{bmatrix}.$$

The variance of the AR(p) process is obtained from Eq. (2.20) for $k = 0$. Now, the expectation $E[Z_t a_t]$ will yield σ_a^2 and the variance is given by

$$\gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 + \cdots + \phi_p\gamma_p + \sigma_a^2. \tag{2.25}$$

Dividing through by $\gamma_0 = \sigma_z^2$ and solving for σ_z^2 results in

$$\sigma_z^2 = \frac{\sigma_a^2}{1 - \phi_1\rho_1 - \phi_2\rho_2 - \cdots - \phi_p\rho_p}. \tag{2.26}$$

It is interesting to study the form of the correlation function ρ_k, which was seen to satisfy

$$\phi(B)\rho_k = 0. \tag{2.27}$$

The polynomial operator $\phi(B)$ can be expanded as

$$\phi(B) = \prod_{i=1}^{p}(1 - G_i B) \tag{2.28}$$

(Box and Jenkins, 1976).

Given Eq. (2.28), the general solution of ρ_k is

$$\rho_k = A_1 G_1^k + A_2 G_2^k + \cdots + A_p G_p^k, \tag{2.29}$$

where $G_1^{-1} \cdots G_p^{-1}$ are the roots of the polynomial (characteristic equation) $\phi(B)$. For stationarity, it was previously seen that $|G_i| < 1$. Following Box and Jenkins, if G_i is real and distinct, then Eq. (2.29) geometrically decays to zero as k increases. If a pair of roots G_i, G_j is complex, they contribute a term

$$d^k \sin(2\pi fk + F)$$

to ρ_k, which is a damped sine wave as k increases.

Before moving ahead, it is important to point out that the AR(p) model has $p + 2$ parameters to be estimated $\{\phi_1, \ldots, \phi_p, \mu, \text{ and } \sigma_a^2\}$.

2.2.1 First-Order Autoregressive Model, AR(1)

The lag-one autoregressive model, or Markov model, is

$$Z_t = \phi_1 Z_{t-1} + a_t$$

or

$$(1 - \phi_1 B)Z_t = a_t$$
$$\phi(B)Z_t = a_t. \tag{2.30}$$

From the Yule–Walker equations,

$$\begin{aligned}\rho_1 &= \phi_1\rho_0\\ &= \phi_1\\ \rho_2 &= \phi_1\rho_1\\ &= \rho_1^2\end{aligned} \tag{2.31}$$

or in general

$$\rho_k = \rho_1^k = \phi_1^k. \tag{2.32}$$

From Eq. (2.26), the variance is

$$\sigma_z^2 = \frac{\sigma_a^2}{1-\phi_1\rho_1} = \frac{\sigma_a^2}{1-\rho_1^2}. \tag{2.33}$$

Stationarity conditions require $|\phi_1| < 1$, which is always the case since $\phi_1 = \rho_1$.

For positive ϕ_1, it is clear that the autocorrelation of AR(1) is of the decaying exponential type. For negative ϕ_1, the autocorrelation will alternate in sign and decay as k increases. This behavior is illustrated in Figs. 2.2a and

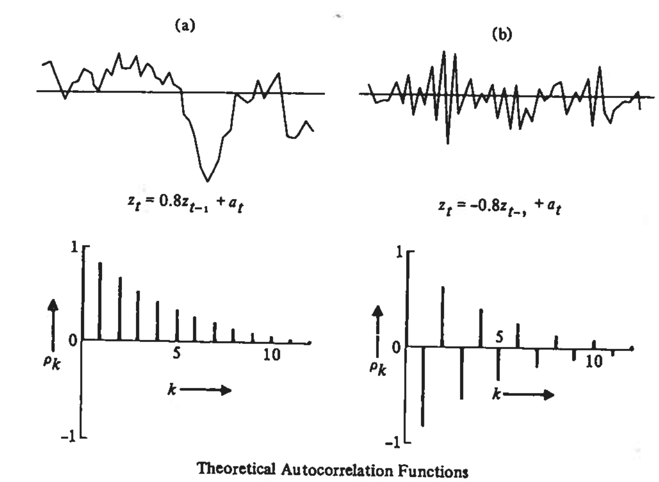

Figure 2.2 Autocorrelation functions of an AR(1) (from Box and Jenkins, 1976).

2.2b. As will be corroborated in Chapter 4, the AR(1) with positive parame is dominated by low-frequency fluctuations while that with negative para ters exhibits dominating high frequencies.

The AR(1) model is the most popular model of time-series simulation forecasting in hydrology and other fields. In hydrology, it is commonly cal the Thomas–Fiering model (Maass et al., 1962). It is usually expressed in te of the moments of the random process X_t,

$$(X_t - \mu) = \rho_1(X_{t-1} - \mu) + \sigma_x(1-\rho_1^2)^{1/2}W_t, \qquad (2.$$

where $X_t - \mu$ is Z_t in the previous paragraphs.

In the above expression, μ is the stationary mean of the process X_t. T random term a_t is substituted by the expression $\sigma_x(1-\rho_1^2)^{1/2}W_t$, where W_t i zero mean, variance 1, normally distributed random variable. The expressi $\sigma_x(1-\rho_1^2)^{1/2}$ is, according to Eq. (2.33), the standard deviation of a_t.

The AR(1) model is frequently used with the sole objective of preservi first- and second-order moments of time series. Such an objective is sometin sufficient for simulation purposes and generally adequate for short-term fo casting. Parameter estimation using the method of moments simply requi substitution of sample moments $\bar{X}$, r_1, and S_x^2 for population moments, μ, and σ_x^2, respectively, in Eq. (2.34). Common equations for these statistics

$$\bar{X} = \frac{1}{N}\sum_{i=1}^{N} x_i \qquad (2.3$$

$$\begin{aligned}S_x^2 &= \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{X})^2\\ &= \frac{1}{N}\sum_{i=1}^{N}x_i^2 - \bar{X}^2\end{aligned} \qquad (2.3$$

$$r_1 = \frac{\frac{1}{N}\sum_{i=1}^{N-1}(x_i - \bar{X})(x_{i+1} - \bar{X})}{S_x^2} \qquad (2.3$$

where the lower case variable x_i denotes sample values of the random var able X.

Keep in mind that the above statistics are random variables and can t expected to vary (some considerably) when estimated from different sample The properties of the above and other estimators are discussed in the statistic literature. Users of synthetic hydrology should be aware of these propertie This opens for discussion the wisdom of fitting models to highly variabl sample statistics. Because of sampling uncertainty, there is no assurance th the moments of the underlying populations are preserved. Synthetic hydrolog and simulation cannot improve basic statistical reliability over historically available information. It only offers the opportunity to experiment with statistically similar realizations. Methods that take into account the stochasticity of the parameters are discussed by Vicens et al. (1975), Valdes et al. (1977), and McLeod and Hipel (1978).

The random term of Eq. (2.34), W_t, has been assumed throughout the previous discussions to be normally distributed. Since the addition of normal variates generates other Gaussian variables, Eq. (2.34) represents a Gaussian process. As such, the process is fully defined with the first- and second-order distributions, both Gaussian. Given a value of X_{t-1}, it is clear that the distribution of X_t, conditional on x_{t-1}, is also normal with mean

$$E[X_t|x_{t-1}] = \mu + \rho_1(x_{t-1} - \mu) \tag{2.38}$$

and variance

$$\sigma^2_{X_t|x_{t-1}} = \sigma_x^2[1 - \rho_1^2]. \tag{2.39}$$

Note that the variance of the process is reduced by $1 - \rho_1^2$ when a previous value is available.

It can be shown that Eqs. (2.38) and (2.39) result from obtaining the conditional distribution using Eq. (1.21) where the joint distribution of consecutive flows is normal and so is the marginal distribution. Since the conditional distribution is also normal, the propagation of means and variances given in the above equations represents an exact solution to the simulation problem as defined in Section 1.3. Sampling from the conditional distribution is automatically achieved by generating the standard normal deviate W_t.

Equations (2.38) and (2.39) also form a prediction model, where the prediction of the state X_t is the conditional mean value, which, for the Gaussian model, happens to be a linear function of previously observed quantities. Theoretically then, a model of the type of Eq. (2.34) should not be used unless there is reason to suspect that the underlying process of interest has approximately linear expressions for the conditional mean. Nevertheless, as mentioned in Chapter 1, many times the interest is in simulation and preservation of finite numbers of moments of the historical observations. If the user is interested in mean, variance, and lag-one correlation, and the process is not highly non-Gaussian, then use of a model of the type given in Eq. (2.34) is adequate.

Hydrologic simulation sometimes requires that third moments or skewness of historical time series be preserved. Statistically, the skewness coefficient is estimated as:

$$\gamma = \frac{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{X})^3}{S_x^3}. \tag{2.40}$$

This is known to be a bounded and highly variable estimator (Kirby, 197 It is therefore questionable whether sample skewness should be preserv Nevertheless, if skewness is desired, the present formulation of Eq. (2.34) inadequate.

Several schemes have been proposed to force skewness on the results autoregressive models. One procedure is to modify the model:

$$X_t = \mu + \rho_1(X_{t-1} - \mu) + \sigma_x(1 - \rho_1^2)^{1/2}\varepsilon_t, \qquad (2.4$$

where ε_t is an approximately gamma-distributed variate with zero mea variance 1, and skewness coefficient γ_ε. Such a model would resemble a proce X_t with mean μ, variance σ_x^2, lag-one correlation ρ_1, and skewness γ_x. T skewness coefficient of the process is related to the skewness coefficient of tl generating deviate ε_t by

$$\gamma_\varepsilon = \frac{1 - \rho_1^3}{(1 - \rho_1^2)^{1.5}}\gamma_x.$$

The εs can be generated with the skewness given by the above equatio using an approximate expression

$$\varepsilon_t = \frac{2}{\gamma_\varepsilon}\left[\left(1 + \frac{\gamma_\varepsilon W_t}{6} + \frac{\gamma_\varepsilon^2}{36}\right)\right]^3 - \frac{2}{\gamma_\varepsilon} \qquad (2.42$$

(Wilson and Hillerty, 1931), where W_t is a normally distributed variable wit zero mean and variance 1.

Equation (2.42) is in fact an approximation of a chi-square-distribute variable, which is theoretically valid for degrees of freedom ν greater than 3(This implies a very small possible skewness coefficient, since $\gamma = \sqrt{8/\nu}$ Nevertheless, some investigators (e.g., Mejia, 1971) claim that the bias intro duced by using Eq. (2.42) in generating random variables with skewnes greater than $\sqrt{8/30} \approx (0.52)$ is acceptable within the range of commonl observed streamflow statistics.

Another method suggested by Yevjevich (1966) and discussed by Matala (1967) involves a three-step procedure to form approximately gamma-distrib uted variables. First, a zero mean, unit variance, lag-one correlation ρ_y proces is formed using an autoregressive model:

$$Y_t = \rho_y Y_{t-1} + (1 - \rho_y^2)^{1/2} W_t. \tag{2.43}$$

A number m of variates Y_t is selected and used to form a new process Z:

$$Z_i = \sum_{j=(i-1)m+1}^{im} Y_j^2. \tag{2.44}$$

Note that Z_i consists of groupings of squares of the original variate Y_j. So, for example, 100 Ys and $m=10$ lead to only 10 values of Z. The variate Z is approximately gamma with mean m, variance $2m$, skewness coefficient $2\sqrt{2/m}$, and lag-one correlation ρ_y^2. By renormalizing the time series of Zs, it is possible to obtain a new series X_t with any desired mean and variance:

$$X_t = \bar{X} + S_x\left[(2m)^{-0.5}Z_t - (m/2)^{0.5}\right]. \tag{2.45}$$

The resulting series of X_ts will have mean $\bar{X}$, variance S_x^2, lag-one correlation ρ_y^2, and skewness coefficient $\gamma_x = 2\sqrt{2/m}$. Clearly, m is a parameter to be determined as a function of the desired skewness. The highest skewness the model can preserve is $2\sqrt{2} \approx 2.8$.

Possibly the most common attempt to preserve skewness is the generation of log-normally distributed variables. Assume that X_t is log-normal, such that

$$Y_t = \ln(X_t - a) \tag{2.46}$$

is a normally distributed variable. The statistics of Y_t and X_t are related through the well-known equations (Matalas, 1967)

$$\mu_x = a + \exp\left[\sigma_y^2/2 + \mu_y\right] \tag{2.47}$$

$$\sigma_x^2 = \exp\left[2(\sigma_y^2 + \mu_y)\right] - \exp\left[\sigma_y^2 + 2\mu_y\right] \tag{2.48}$$

$$\gamma_x = \frac{\exp(3\sigma_y^2) - 3\exp(\sigma_y^2) + 2}{\left[\exp(\sigma_y^2) - 1\right]^{3/2}} \tag{2.49}$$

$$\rho_x = \frac{\exp\left[\sigma_y^2\rho_y\right] - 1}{\exp(\sigma_y^2) - 1}. \tag{2.50}$$

The statistics μ_y, σ_y, and ρ_y, directly computed from the transformed historical sequences, could be used in an autoregressive formulation leading to the generation of normal variates, which, upon exponentiation would yield a log-normal process. Nevertheless, such a procedure will preserve the sample statistics of the transformed historical sequences but produce a biased estimate of the sample statistics of the original variables. Furthermore, the generation of transformed variables will not necessarily preserve the historical skewness of the process. The procedure is so simple, though, that the practitioner many times settles on slightly biased results.

If the unbiased preservation of the sample statistics of the original variables is the goal, then Eqs. (2.47) through (2.50) must be used. The idea is to estimate the parameters μ_x, σ_x, γ_x, and ρ_x of the original variables and simultaneously solve Eqs. (2.47) through (2.50) for the corresponding statistics a, σ_y, μ_y, and ρ_y. The latter statistics are used in developing an autoregressive

model, which, in turn, is used to generate a normally distributed seri $i=1,\ldots,n$. Transforming Y_i

$$X_i = \exp(Y_i) + a$$

results in a sequence X_i with the historical sample statistics μ_x, σ and ρ_x.

Authors such as Burges et al. (1975), Stedinger (1980), and Bobé Robitaille (1975) have extensively discussed the value of the log-n transformation in hydrology. Their discussions of estimation bias and behavior with small data samples may be of interest to the reader.

EXAMPLE 2.1 (adapted from notes by John C. Schaake, Jr.)

Following are 10 years of observations of annual streamflows in millio cubic meters:

Year	Discharge	Year	Discharge
1	145.78	6	175.02
2	95.43	7	101.98
3	116.66	8	146.14
4	96.12	9	126.01
5	122.09	10	132.73

Use the above data to estimate (by the method of moments) the pa ters of an AR(1) model that preserves first, second, and third mor Generate 40 years of synthetic annual streamflows from the resulting mc

Relevant sample moments are obtained from the given data as

$$\bar{Q} = \frac{1}{10}\sum_{i=1}^{10} Q_i = 125.80$$

$$S = \left(\frac{1}{10}\sum_{i=1}^{10} Q_i^2 - \bar{Q}^2\right)^{1/2} = 25.3$$

$$r_1 = \frac{\frac{1}{10}\sum_{i=1}^{10} Q_i Q_{i+1} - \bar{Q}^2}{S^2} = -0.28.$$

(*Note:* in the above, $Q_{11} = Q_1$; this is a circular definition of correlation. an unusual negative correlation, probably due to the small sample used i estimation, is observed.)

$$\gamma = \frac{\frac{1}{10}\sum_{i=1}^{10} (Q_i - \bar{Q})^3}{S^3} = 0.42.$$

Following are three sets of 40 years of simulated streamflows and corresponding summary statistics. The first column corresponds to streamflows generated by taking the logarithms of the original data, generating normally distributed variables, and then exponentiating the results. Column 2 uses the gamma approximation given in Eq. (2.41). Column 3 used the full log transformation as described in Eqs. (2.46) through (2.51).

Run 1 ($\times 10^2$)	Run 2 ($\times 10^2$)	Run 3 ($\times 10^2$)
1.25796	1.25796	1.25796
1.04610	1.14337	1.23635
1.45375	1.53361	1.61603
1.09044	1.14088	1.72124
1.10708	1.06840	1.06668
1.37060	1.27093	1.33715
1.18291	1.38975	1.44948
1.08402	1.36419	1.31257
1.38411	0.81018	1.23944
1.03360	0.98727	1.30080
1.57609	1.81578	1.31322
0.81442	1.34882	0.77345
1.90717	1.09977	1.94454
0.97410	1.65840	1.01594
1.33417	1.09074	1.04225
0.81967	1.30820	1.27225
1.22386	1.63686	1.10791
1.21820	0.93435	1.30460
0.65722	1.44445	1.33684
1.30744	0.84896	1.24458
1.48888	1.32676	1.08977
1.39215	1.64438	1.45393
1.31445	1.03494	1.23296
1.31683	1.45548	1.38069
1.16038	0.89716	1.32512
1.37676	1.71125	1.14234
1.36500	0.83465	1.33604
1.53774	1.34727	1.25665
1.41753	1.54085	1.12521
1.52989	1.41571	1.05186
1.38218	1.21430	1.28745
1.42435	1.78309	1.18783
1.41133	0.87965	1.42316
1.29935	1.84489	0.82781
1.10013	1.01661	1.41398
1.40968	1.17430	1.07725
1.20144	1.75676	1.36271
1.25377	0.89639	0.76215

(cont'd)

	Run 1 ($\times 10^2$)	Run 2 ($\times 10^2$)	Run 3 ($\times 10^2$)
	1.03618	1.06355	1.36016
	0.49733	1.86203	1.10057
Mean	124.4	129.6	125.2
Standard deviation	26.0	31.4	22.7
Lag-one correlation	−0.17	−0.47	−0.36
Skewness coefficient	−0.56	0.22	0.33

2.2.2 Second-Order Autoregressive Model

Second order is usually the highest lag necessary in representing hydrc time series. The model takes the form

$$\phi(B)Z_t = a_t$$
$$(1-\phi_1 B - \phi_2 B^2)Z_t = a_t. \quad ($$

The more familiar representation is

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + a_t. \quad ($$

For stationarity, the roots (solutions of B) of the quadratic polyn $\phi(B)$ must lie outside the unit circle. Using the well-known solutio quadratic equations, it can be shown that the implied conditions on paran ϕ_1 and ϕ_2 are

$$\phi_2 + \phi_1 < 1$$
$$\phi_2 - \phi_1 < 1 \quad ($$
$$-1 < \phi_2 < 1.$$

Figure 2.3 shows the triangular parameter space defined by Eq. (2.54). The Yule–Walker equations for the AR(2) model are

$$\rho_1 = \phi_1 + \phi_2 \rho_1$$
$$\rho_2 = \phi_1 \rho_1 + \phi_2, \quad ($$

which when solved simultaneously yield

$$\phi_1 = \rho_1(1-\rho_2)/(1-\rho_1^2)$$
$$\phi_2 = \frac{\rho_2 - \rho_1^2}{1-\rho_1^2}, \quad ($$

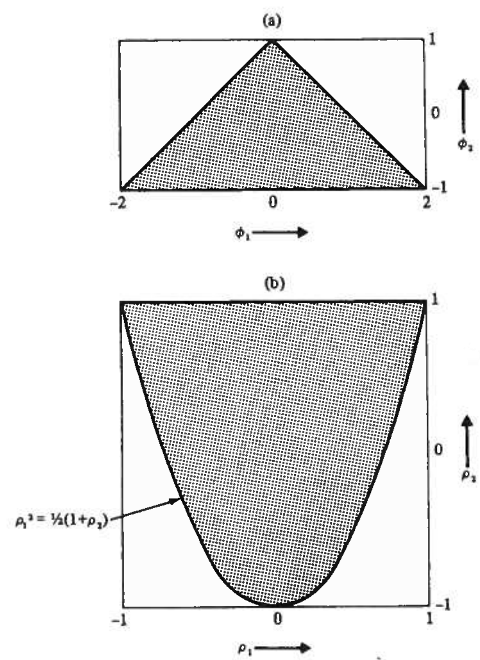

Figure 2.3 Valid regions of the parameters and correlations of a stationary AR(2) process (from Box and Jenkins, 1976).

or inversely,

$$\rho_1 = \frac{\phi_1}{1-\phi_2}$$
$$\rho_2 = \phi_2 + \frac{\phi_1^2}{1-\phi_2}. \qquad (2.57)$$

Figure 2.4 gives the solution of Eq. (2.56) for various values of correlation coefficients ρ_1 and ρ_2. In practice, sample estimates could be used for the correlations in order to obtain parameter values.

The parameter limits given in Eq. (2.54) and the relations between correlations and parameters given in Eq. (2.57) define a region of valid correlation

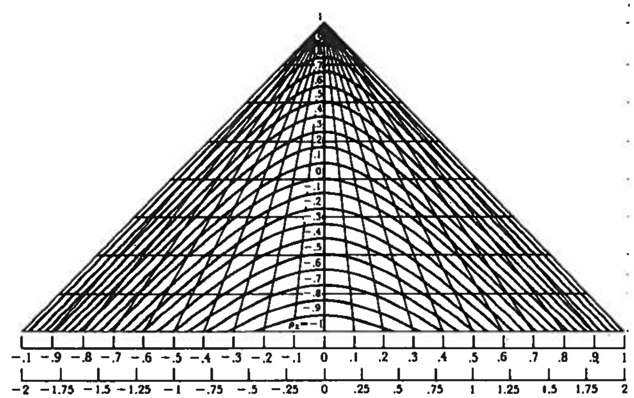

Figure 2.4 Relation between correlations and parameters of an AR(2) moc Diagram may be used for parameter estimation using the method of momen (From Box and Jenkins, 1976.)

coefficients for the lag-two (second-order) autoregressive model,

$$-1 < \rho_1 < 1$$
$$-1 < \rho_2 < 1 \qquad ($$
$$\rho_1^2 < \tfrac{1}{2}(\rho_2 + 1).$$

The admissible regions of parameters and correlations are shown i 2.3. Notice that this figure could be used as a first-cut criteria for the po use of an AR(2) with a given set of data.

The correlation function of the AR(2) model is easily obtained by sively solving

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} \qquad ($$

with initial conditions,

$$\rho_0 = 1$$
$$\rho_1 = \frac{\phi_1}{1-\phi_2}.$$

Box and Jenkins (1976) give the general solution to Eq. (2.59) as

$$\rho_k = \frac{G_1(1-G_2^2)G_1^k - G_2(1-G_1^2)G_2^k}{(G_1-G_2)(1+G_1G_2)},$$

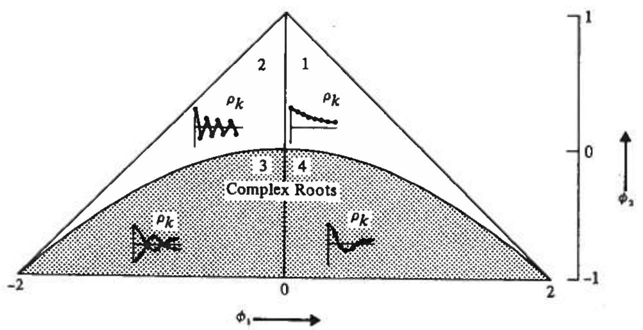

Figure 2.5 Autocorrelation functions of various stationary AR(2) models (from Box and Jenkins, 1976).

where G_1^{-1} and G_2^{-1} are the roots of $\phi(B)$. Real roots result when $\phi_1^2 + 4\phi_2 \geq 0$. Complex roots are obtained when the discriminant is less than 0, $\phi_1^2 + 4\phi_2 < 0$. Regions 1 and 2 of Fig. 2.5 illustrate the behavior of the correlation for real roots—Region 1 for a dominant positive root and Region 2 for a negative dominant root. The correlation decays geometrically either monotonically (Region 1) or alternating signs (Region 2). Conjugate complex roots result in a damped sinusoid with a phase angle between 90 degrees and 180 degrees in Region 4.

Using Eq. (2.26), the variance of the AR(2) model is

$$\sigma_z^2 = \frac{\sigma_a^2}{1 - \rho_1\phi_1 - \rho_2\phi_2} = \left(\frac{1-\phi_2}{1+\phi_2}\right)\frac{\sigma_a^2}{\left[(1-\phi_2)^2 - \phi_1^2\right]}. \tag{2.60}$$

EXAMPLE 2.2

The autocorrelation of the St. Lawrence River at Ogdensburg, New York, as given by Yevjevich (1972) and McLeod et al. (1977), is shown in Fig. 2.6.

The lag-one autocorrelation is observed at around 0.7 and the lag-two value is about 0.5. The mean discharge is 6825 m^3/sec with standard deviation of 544 m^3/sec and skewness coefficient of -0.286 (Yevjevich, 1963).

A lag-one autoregressive model of the St. Lawrence fitted by the method of moments would be

$$Z_t = 6825 + 0.7(Z_{t-1} - 6825) + 544(1 - 0.7^2)^{1/2} W_t$$
$$= 6825 + 0.7(Z_{t-1} - 6825) + 388.49 W_t.$$

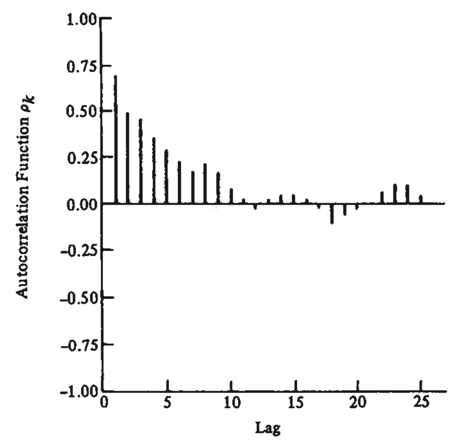

Figure 2.6 Autocorrelation function for the Saint Lawrence River (from McLeod, et al. in *Water Resources Research* 13(3):577-86, 1977).

The lag-one and lag-two correlations are within the admissible regions Fig. 2.3. Using Eq. (2.56) (or Fig. 2.4) the method of moments fit of an AR model would yield the following parameters

$$\phi_1 = r_1(1 - r_2)/(1 - r_1^2) = 0.69$$
$$\phi_2 = \frac{r_2 - r_1^2}{1 - r_1^2} = 0.02.$$

Equation (2.60) yields

$$S_a^2 = S^2(1 - \rho_1\phi_1 - \rho_2\phi_2)$$
$$= 295{,}936[1 - 0.7(0.69) - 0.5(0.02)] = 150{,}040$$

or

$$S_a = 387.$$

Therefore an estimated AR(2) would be

$$Z_t = 6825 + 0.69(Z_{t-1} - 6825) + 0.02(Z_{t-2} - 6825) + 387 W_t.$$

Given the inherent variability in moment estimation the reader should suspect the statistical significance of ϕ_2.

2.3 MOVING-AVERAGE MODELS OF ORDER q, MA(q)

As in the autoregressive models, moving-average formulations of infinite order have no practical use. A finite-order model is represented by

$$\begin{aligned} Z_t &= a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} \\ &= (1 - \theta_1 B - \cdots - \theta_q B^q) a_t \\ &= \theta(B) a_t. \end{aligned} \tag{2.61}$$

Since the polynomial $\theta(B)$ is finite, it always converges, implying unconditional stationarity, according to the criteria discussed in Section 2.1.1. The invertibility of Eq. (2.61) requires $\theta^{-1}(B)$ to converge for $|B| \le 1$. Box and Jenkins (1976) show that this is equivalent to stating that the roots of the characteristic equation $\theta(B)$ must lie outside the unit circle. It is important to note that a finite moving-average model is equivalent to an infinite autoregressive model.

The autocovariance function of the MA(q) process is obtained by performing the following expectation:

$$\gamma_k = E\left[(a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q})(a_{t-k} - \theta_1 a_{t-k-1} - \cdots - \theta_q a_{t-k-q})\right].$$

For $k = 0$ and using the "whiteness" properties of the series a_k, the process variance becomes

$$\gamma_0 = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma_a^2. \tag{2.62}$$

For $k \ne$ zero, it is clear that the autocovariance is

$$\gamma_k = \begin{cases} (-\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_{q-k}\theta_q)\sigma_a^2 & k = 1, 2, \ldots, q \\ 0 & k > q. \end{cases}$$

Upon normalizing the autocovariance by γ_0 the autocorrelation fun[...] becomes

$$\rho_k = \begin{cases} \dfrac{-\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \cdots + \theta_q^2} & k = 1, 2, \ldots, q \\ 0 & k > q. \end{cases} \quad (\text{[...]}$$

Therefore the autocorrelation of a moving-average model is zero beyon[...] order of the model. This is in contrast to the infinite extent of ρ_k in the A[...] model.

The nonlinearity of the autocorrelation expressions makes paran[...] estimation for the MA(q) model using the method of moments highly u[...]ble. Iterative least-squares procedures are usually required (Box and Jen[...] 1976).

2.3.1 First-Order Moving-Average Model, MA(1)

The form of the MA(1) is

$$\begin{aligned} Z_t &= a_t - \theta_1 a_{t-1} \\ &= (1 - \theta_1 B) a_t, \end{aligned} \quad (\text{[...]}$$

which is stationary for all values of θ_1 but invertible only for $|B| > 1$, w[...] implies (since $B = \theta^{-1}$) that $|\theta_1| < 1$.

The variance of the process is

$$\gamma_0 = (1 + \theta_1^2)\sigma_a^2 \quad (\text{[...]}$$

and its autocorrelation function

$$\rho_k = \begin{cases} \dfrac{-\theta_1}{1 + \theta_1^2} & k = 1 \\ 0 & k \ge 2, \end{cases} \quad (\text{[...]}$$

which (for $k = 1$) leads to the following relation between θ_1 and ρ_1

$$\theta_1^2 + \frac{\theta_1}{\rho_1} + 1 = 0. \quad (\text{[...]}$$

Note that the autocorrelation disappears after lag one. The two roots of (2.67) must then be θ_1 and θ_1^{-1}. The invertibility conditions will be satisfied i[...] root $|\theta_1| < 1$ is used.

Table 2.1
Table relating ρ_1 to θ for a first-order moving-average process

θ	ρ_1	θ	ρ_1
0.00	0.000	0.00	0.000
0.05	−0.050	−0.05	0.050
0.10	−0.099	−0.10	0.099
0.15	−0.147	−0.15	0.147
0.20	−0.192	−0.20	0.192
0.25	−0.235	−0.25	0.235
0.30	−0.275	−0.30	0.275
0.35	−0.315	−0.35	0.315
0.40	−0.349	−0.40	0.349
0.45	−0.374	−0.45	0.374
0.50	−0.400	−0.50	0.400
0.55	−0.422	−0.55	0.422
0.60	−0.441	−0.60	0.441
0.65	−0.457	−0.65	0.457
0.70	−0.468	−0.70	0.468
0.75	−0.480	−0.75	0.480
0.80	−0.488	−0.80	0.488
0.85	−0.493	−0.85	0.493
0.90	−0.497	−0.90	0.497
0.95	−0.499	−0.95	0.499
1.00	−0.500	−1.00	0.500

Source: Box and Jenkins (1976).

Table 2.1 gives the invertible solution for Eq. (2.67) for lag−1 correlation between −0.5 and 0.5 (Box and Jenkins, 1976).

2.3.2 Second-Order Moving-Average Model, MA(2)

The MA(2) model is given by

$$\begin{aligned} Z_t &= a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} \\ &= \left(1 - \theta_1 B - \theta_2 B^2\right) a_t \\ &= \theta(B) a_t. \end{aligned} \qquad (2.68)$$

Since the polynomial $\theta(B)$ converges for $|B| \leq 1$, the model is unconditionally stationary. Invertibility requires that the roots of $\theta(B)$ lie outside the unit circle. This is analogous to the stationarity conditions of the AR(2) model

given by Eq. (2.54). The implied parameter space is then the same:

$$\begin{aligned} \theta_2 + \theta_1 &< 1 \\ \theta_2 - \theta_1 &< 1 \\ -1 < \theta_2 &< 1. \end{aligned} \qquad ($$

From Eq. (2.62), the variance is

$$\gamma_0 = \sigma_a^2\left(1 + \theta_1^2 + \theta_2^2\right). \qquad (:$$

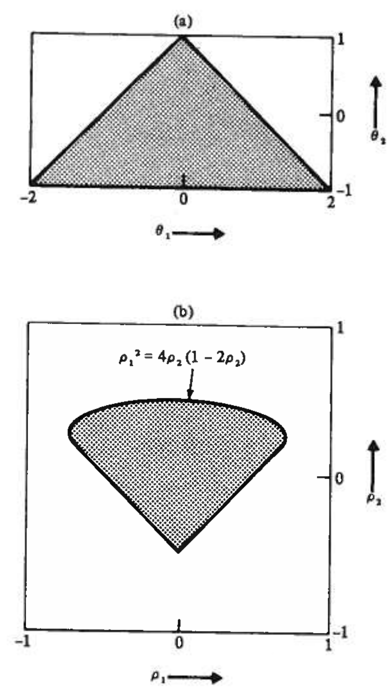

Figure 2.7 Valid regions of the parameters and the correlations of an invert MA(2) process (from Box and Jenkins, 1976).

The autocorrelation function, Eq. (2.63), is

$$\rho_1 = \frac{-\theta_1(1-\theta_2)}{1+\theta_1^2+\theta_2^2} \qquad \rho_2 = \frac{-\theta_2}{1+\theta_1^2+\theta_2^2} \qquad (2.71)$$
$$\rho_k = 0 \qquad k \geq 3.$$

Invertibility conditions on parameters and their explicit relationship with ρ_1 and ρ_2 force the following limits on correlations of MA(2) models

$$\rho_2 + \rho_1 = -0.5$$
$$\rho_2 - \rho_1 = -0.5 \qquad (2.72)$$
$$\rho_1^2 = 4\rho_2(1-2\rho_2).$$

Figure 2.7 gives the invertible-parameter region and the limits on correlations imposed by the second-order moving-average model. Figure 2.8 is the solution to the nonlinear Eq. (2.71), giving θ_1 and θ_2 as a function of ρ_1 and ρ_2. Sample estimates of the correlations can then be used to obtain initial estimates of the parameter values. Again, the variability of sample correlation estimates and the nonlinearity of the relationship between moments and parameters may lead to serious problems in parameter estimation.

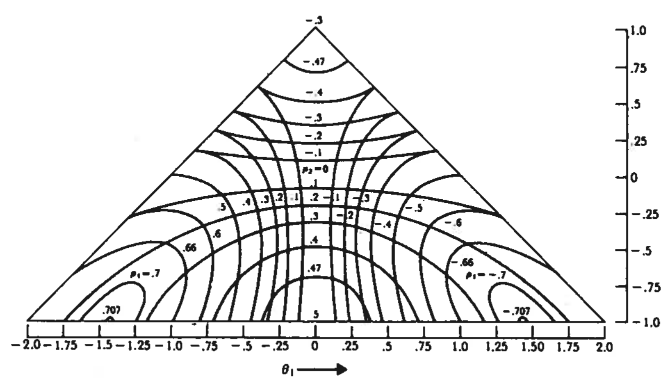

Figure 2.8 Relation between correlations and parameters for a second-order moving-average model. Diagram may be used for parameter estimation using the method of moments. (From Box and Jenkins, 1976.)

2.4 AUTOREGRESSIVE–MOVING-AVERAGE MODELS, ARMA(p, q)

Autoregressive and moving-average models can be combined to model proc that otherwise would be operationally impossible to represent with single AR or MA models. An ARMA(p, q) model takes the form

$$\left(1-\phi_1 B-\phi_2 B^2-\cdots-\phi_p B^p\right) Z_t=\left(1-\theta_1 B-\theta_2 B^2-\cdots-\theta_q B^q\right) a_t$$
$$\phi(B) Z_t=\theta(B) a_t. \qquad ($$

The stationarity and invertibility conditions of the ARMA(p, q) m correspond to those of the component MA and AR models. For station and invertibility, the roots of $\phi(B)$ and of $\theta(B)$ must lie outside the unit ci

The autocovariance function is found by multiplying $Z_t = \phi_1 \cdot + \cdots + \phi_p Z_{t-p} + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$ by Z_{t-k} and finding expe values,

$$\gamma_k = E[Z_t Z_{t-k}] = \phi_1\gamma_{k-1} + \cdots + \phi_p\gamma_{k-p} + \gamma_{za}(k)$$
$$-\theta_1\gamma_{za}(k-1) - \cdots - \theta_q\gamma_{za}(k-q), \qquad (2$$

where

$$\gamma_{za}(k) = E[Z_{t-k}a_t]$$
$$\gamma_{za}(k-1) = E[Z_{t-k}a_{t-1}]. \qquad (2$$

The value for $\gamma_{za}(k)$ will be zero as long as $k>0$, since no correlation e between a_t and values of Z before t. The value for $\gamma_{za}(k)$ will not be zero for k

With the above in mind, it should be clear that for $k > q$, the autoco ance (and autocorrelation) in Eq. (2.74) reduces to that of an AR(p) mc

$$\gamma_k = \phi_1\gamma_{k-1} + \cdots + \phi_p\gamma_{k-p} \qquad \text{for } k > q$$
$$\rho_k = \phi_1\rho_{k-1} + \cdots + \phi_p\rho_{k-p} \qquad \text{for } k > q.$$

For values of k less than or equal to q, the autocovariance will t function of the moving-average terms and will depend on all coeffici $\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q$, and the variance σ_a^2. The ARMA(p, q) model then has convenient property that its first q autocorrelations depend on moving-ave terms as well as autoregressive terms. After q lags, autoregressive beha takes over from the last correlation value.

The variance of the process is given by Eq. (2.74) for $k = 0$. Evaluatio the variance requires the solution of $\gamma_1, \ldots, \gamma_p$.

2.4.1 ARMA(1,1)

A popular, and useful, model in hydrology is

$$Z_t - \phi_1 Z_{t-1} = a_t - \theta_1 a_{t-1}$$
$$(1-\phi_1 B)Z_t = (1-\theta_1 B)a_t. \tag{2.76}$$

Stationarity and invertibility conditions correspond to the individual AR(1) and MA(1) models and so imply that the parameter region is

$$-1 < \phi_1 < 1$$
$$-1 < \theta_1 < 1.$$

Figure 2.9 shows this admissible parameter space.

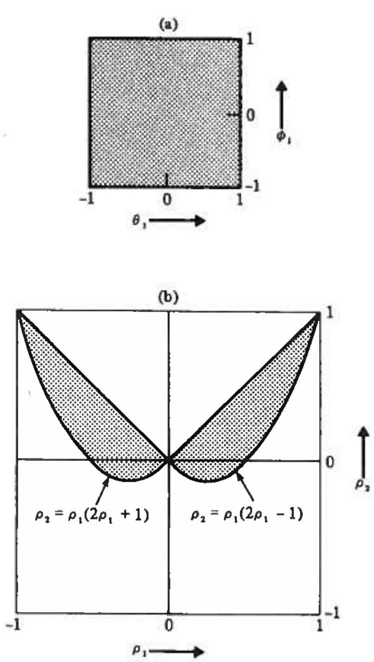

Figure 2.9 Valid regions for the parameters and correlations of a stationary and invertible ARMA(1,1) process (from Box and Jenkins, 1976).

Using Eq. (2.74), the autocovariance function is

$$\gamma_0 = \phi_1\gamma_1 + \sigma_a^2 - \theta_1\gamma_{za}(-1)$$
$$\gamma_1 = \phi_1\gamma_0 - \theta_1\sigma_a^2 \tag{2.77}$$
$$\gamma_k = \phi_1\gamma_{k-1} \qquad k \geq 2.$$

To obtain $\gamma_{za}(-1)$ Eq. (2.76) is multiplied by a_{t-1} and expectations are taken:

$$\gamma_{za}(-1) = E[Z_t a_{t-1}] = (\phi_1 - \theta_1)\sigma_a^2. \tag{2.78}$$

Using Eq. (2.78) in Eq. (2.77) the autocovariance function of the process is obtained as

$$\gamma_0 = \frac{1+\theta_1^2 - 2\phi_1\theta_1}{1-\phi_1^2}\sigma_a^2$$
$$\gamma_1 = \frac{(1-\phi_1\theta_1)(\phi_1-\theta_1)}{1-\phi_1^2}\sigma_a^2 \tag{2.79}$$
$$\gamma_k = \phi_1\gamma_{k-1} \qquad k \geq 2.$$

Note that the autocovariance will decay exponentially from a starting value γ_1, which is dependent on θ_1. The sign of γ_1 (and ρ_1) is defined by $\phi_1 - \theta_1$. The sign of ϕ_1 determines if the correlation decay is smooth or alternates in sign.

The correlation function is given by

$$\rho_1 = \frac{(1-\phi_1\theta_1)(\phi_1-\theta_1)}{1+\theta_1^2 - 2\phi_1\theta_1} \tag{2.80}$$
$$\rho_k = \phi_1\rho_{k-1} \qquad k \geq 2.$$

The relationships shown in Eq. (2.80) and the invertibility–stationarity parameter space define an admissible region for the first two correlations

$$|\rho_2| < |\rho_1|$$
$$\rho_2 > \rho_1(2\rho_1 + 1) \qquad \rho_1 < 0 \tag{2.81}$$
$$\rho_2 > \rho_1(2\rho_1 - 1) \qquad \rho_1 > 0.$$

Figure 2.9 illustrates the above region; correlations outside that space indicate that the ARMA(1,1) is not a good model. Figure 2.10 diagrams the solution of parameter ϕ_1 and θ_1 in terms of ρ_1 and ρ_2 as given by Eq. (2.80). Figure 2.11 gives typical forms of the autocorrelation expected for various regions of the parameter space.

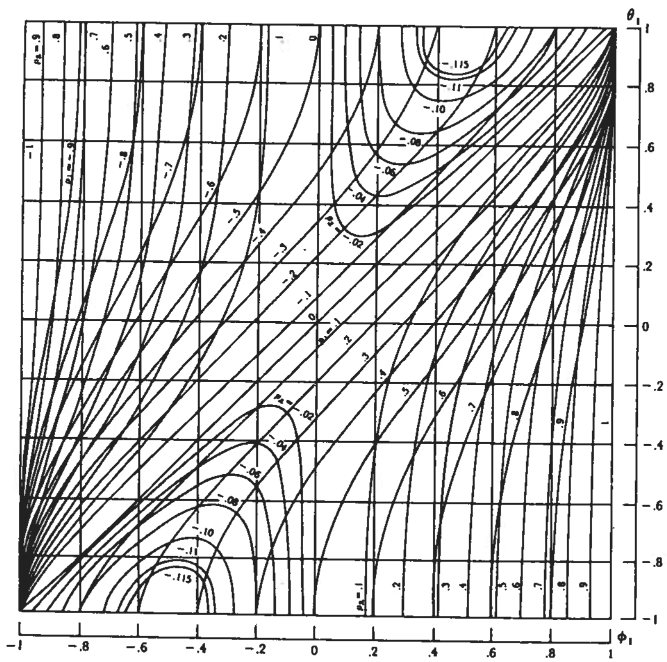

Figure 2.10 Relation between correlations and parameters for a stationary and invertible ARMA(1, 1). Diagram may be used for parameter estimation using the method of moments. (From Box and Jenkins, 1976).

EXAMPLE 2.3

Figure 2.12 gives the sample autocorrelation of the Niger River annual streamflows, which are shown in Fig. 2.13. The observed lag-one autocorrelation r_1 is 0.554 and the lag-two autocorrelation r_2 is around 0.45. Clearly, the rate of correlation decay is much slower than that of an AR(1) or MA(1) model. The autocorrelation does fall relatively quickly for higher lags. Carlson et al. (1970), following identification and estimation procedures discussed in Section 2.6, concluded that an ARMA(1,1) model provided the best fit to the data. They suggested that

$$Z_t = 0.82Z_{t-1} + a_t - 0.4a_{t-1}.$$

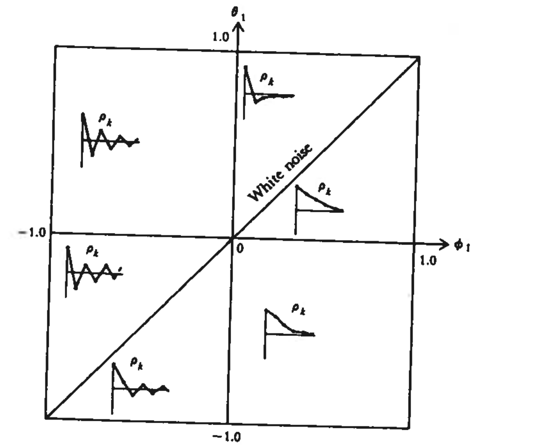

Figure 2.11 Autocorrelation functions for various ARMA(1, 1) models (from Box and Jenkins, 1976).

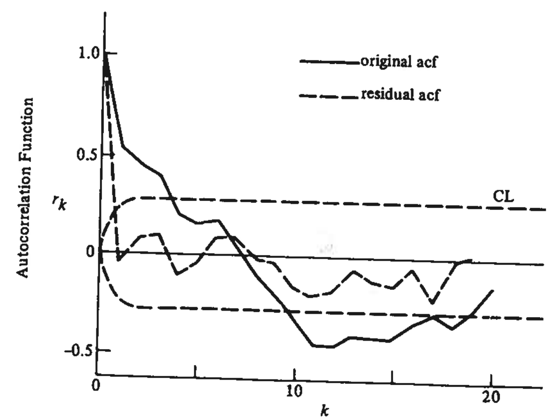

Figure 2.12 Sample autocorrelation function (acf) of Niger data series and the autocorrelation function of the residual series from best fit model for Niger series. The approximate 95% confidence limits (CL) of the autocorrelation function of the residual series are indicated by dashed lines. (From Carlson et al. in *Water Resources Research* 6(4):1070-8, 1970.)

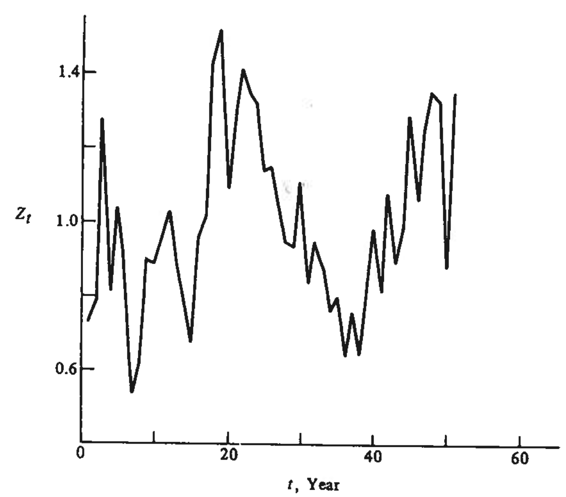

Figure 2.13 Flow of Niger River at Koulicoro, Africa, 1906–1957. Z_t is given as a modular coefficient, and t is time from beginning of series. (From Carlson et al. in *Water Resources Research* 6(4):1070-8, 1970).

The coefficients were obtained using the least-squares procedures discussed in Section 2.6. Using the method of moments and Fig. 2.10 yield

$$Z_t = 0.8Z_{t-1} + a_t - 0.4a_{t-1}.$$

2.5 NONSTATIONARY MODELS: THE AUTOREGRESSIVE INTEGRATED MOVING-AVERAGE MODEL (ARIMA)

2.5.1 Local Nonstationarities

A relatively simple extension of the theory for ARMA processes permits the handling of a limited type of nonstationary model, essentially nonstationarities in the mean of the process.

In previous sections, it was seen that the model

$$\psi(B)Z_t = \theta(B)a_t \tag{2.82}$$

is stationary if the roots of $\psi(B)$ are outside the unit circle. Assume now that

some of the roots of $\psi(B)$ lie on the unit circle. The model can then expressed as

$$\psi(B)Z_t = \phi(B)(1-B)^d Z_t = \theta(B)a_t \qquad (2$$

or

$$\phi(B)\nabla^d Z_t = \theta(B)a_t,$$

where the roots of $\phi(B)$ are outside the unit circle. Because of the differen operation, ∇^d, $(d \geq 1)$, the model can be reformulated in terms of the non mean process X_t, since $\nabla^d Z_t = \nabla^d X_t$. So the above is equivalent to $\phi(B)\nabla$ $= \theta(B)a_t$, where the roots of $\phi(B)$ are outside the unit circle. Defining

$$Y_t = \nabla^d X_t, \qquad (2$$

Eq. (2.83) represents a stationary–invertible ARMA process on the variabl Given Y_t, X_t can be obtained by performing the summation operation,

$$X_t = S^d Y_t, \qquad (2$$

where

$$SY_t = \sum_{h=-\infty}^{t} Y_h$$

$$S^2Y_t = \sum_{i=-\infty}^{t} \sum_{h=-\infty}^{i} Y_h$$

and so on.

Since X_t is then essentially the integration of an ARMA model, particular formulation is called the autoregressive integrated moving averag orders p, d, and q [ARIMA(p,d,q)], where d is the order of differentia of the original data necessary to obtain a stationary process. Note that,

$$\text{ARMA}(p,q) = \text{ARIMA}(p,0,q)$$
$$\text{AR}(p) = \text{ARIMA}(p,0,0)$$
$$\text{MA}(q) = \text{ARIMA}(0,0,q).$$

It should be clear that ARIMA models handle particular types of r stationarities. By taking first-order differences, it is possible to elimi unknown stochastic biases in the data. The differenced data should be tionary if the behavior was otherwise homogeneous. If a process exhi random changes in slope and level but is otherwise homogeneous, t differencing it twice will yield a stationary process.

Deterministic linear trends or higher-order polynomial trends can be incorporated by stating the ARIMA model as

$$\phi(B)\nabla^d X_t = \theta_0 + \theta(B)a_t. \tag{2.86}$$

ARIMA models can be expressed in three different forms. The first is a difference equation by expanding the polynomial

$$\psi(B) = \phi(B)(1-B)^d = 1 - \psi_1 B - \psi_2 B^2 - \cdots - \psi_{p+d}B^{p+d}$$

so that

$$X_t = \psi_1 X_{t-1} + \cdots + \psi_{p+d}X_{t-p-d} - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} + a_t. \tag{2.87}$$

For example, ARIMA(1,1,1),

$$(1-\phi B)(1-B)X_t = (1-\theta B)a_t,$$

is equivalent to

$$\{1-(1+\phi)B + \phi B^2\}X_t = (1-\theta B)a_t.$$

The model can be expressed as a function of past random shocks a_t,

$$X_t = \Omega(B)a_t. \tag{2.88}$$

Operating with $\psi(B)$ on both sides of the above equation,

$$\psi(B)X_t = \psi(B)\Omega(B)a_t = \theta(B)a_t, \tag{2.89}$$

which indicates that

$$\psi(B)\Omega(B) = \theta(B). \tag{2.90}$$

The weights Ω_j are then obtained by equating coefficients of B in the two polynomials of Eq. (2.90):

$$\begin{aligned}&(1-\psi_1 B - \cdots - \psi_{p+d}B^{p+d})(1+\Omega_1 B + \Omega_2 B^2 + \cdots)\\ &= (1-\theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q).\end{aligned} \tag{2.91}$$

For example, in the previously seen ARIMA(1,1,1)

$$\begin{aligned}\psi(B) &= 1-(1+\phi)B + \phi B^2\\ \theta(B) &= 1-\theta B.\end{aligned}$$

Substituting in Eq. (2.90) and equating coefficients of B yields

$$\begin{aligned}\Omega_0 &= A_0 + A_1 = 1\\ \Omega_1 &= A_0 + A_1\phi\\ \Omega_2 &= A_0 + A_1\phi^2\\ &\vdots\\ \Omega_j &= A_0 + A_1\phi^j,\end{aligned}$$

where

$$A_0 = \frac{1-\theta}{1-\phi}$$
$$A_1 = \frac{\theta-\phi}{1-\phi}.$$

So the ARIMA(1,1,1) model is equivalent:

$$X_t = \sum_{j=0}^{\infty}(A_0 + A_1\phi^j)a_{t-j}. \quad (2$$

Finally, the ARIMA model can be expressed in terms of previous Xs the current shock a_t. Starting with the general model

$$\psi(B)X_t = \theta(B)a_t \quad (2$$

the goal is to obtain,

$$\pi(B)X_t = a_t, \quad (2$$

where

$$\pi(B) = \left(1 - \sum_{j=1}^{\infty}\pi_j B^j\right).$$

Using Eq. (2.93) in Eq. (2.94) yields

$$\psi(B)X_t = \theta(B)\pi(B)X_t \quad (2$$

or

$$\begin{aligned}&(1-\psi_1 B - \cdots - \psi_{p+d}B^{p+d})\\ &= (1-\theta_1 B - \cdots - \theta_q B^q)(1-\pi_1 B - \pi_2 B^2 - \cdots) \quad (2\end{aligned}$$

The coefficients π_j are obtained by equating coefficients of B^j. It is easy to corroborate that the coefficients π_j must add up to 1 for $d \geq 1$.

Again, using the ARIMA(1,1,1) as an example, Eq. (2.96) takes the form

$$\begin{aligned}\pi(B) &= \psi(B)\theta^{-1}(B)\\ &= \left[1-(1+\phi)B+\phi B^2\right](1+\theta B+\theta^2 B^2+\cdots)\end{aligned}$$

resulting in

$$\begin{aligned}\pi_1 &= \phi+(1-\theta),\\ \pi_2 &= (\theta-\phi)(1-\theta),\\ &\vdots \qquad \vdots\\ \pi_j &= (\theta-\phi)(1-\theta)\theta^{j-2} \qquad j \geq 3.\end{aligned} \tag{2.97}$$

2.5.2 Seasonal Nonstationarities

Hydrologic time series of time scales less than a year usually exhibit strong seasonal variability or nonstationarity. The nonstationarity of monthly

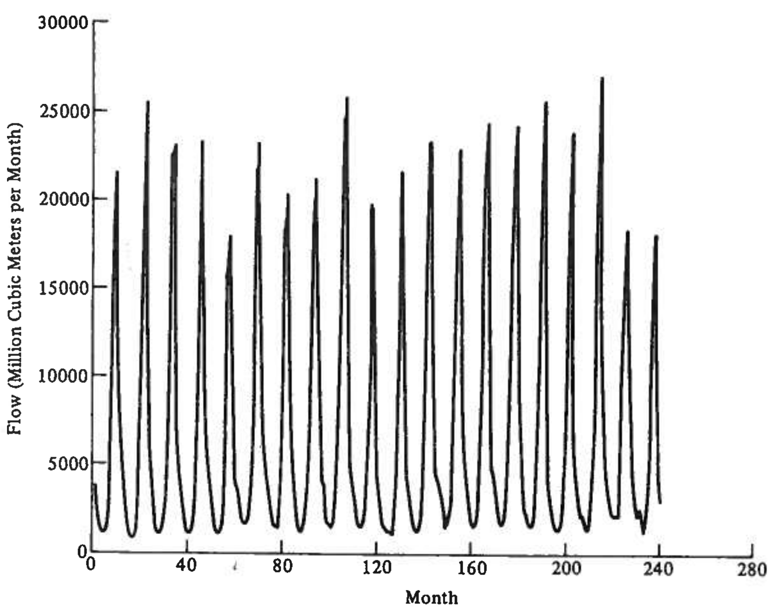

Figure 2.14 Wadi Halfa Streamflow (1921–1940).

streamflows and rainfall volumes in the Nile basin are evident in Figs. 2.15. Figure 2.16 gives the correlation of monthly streamflows of t Saskatchewan River (computed around the annual mean). The re recognize the nondecaying nature of the correlation function, with sin correlations every 12 months. Such behavior represents a particula nonstationarity, where

$$X_t = X_{t-s} + \varepsilon_t,$$

and s is the basic period of the nonstationarity.

Equation (2.98) is of the form,

$$(1-B^s)X_t = \varepsilon_t,$$

where the operator B^s implies

$$B^s X_t = X_{t-s},$$

and ε_t is a residual with undefined structure. In fact, ε_t can be mod

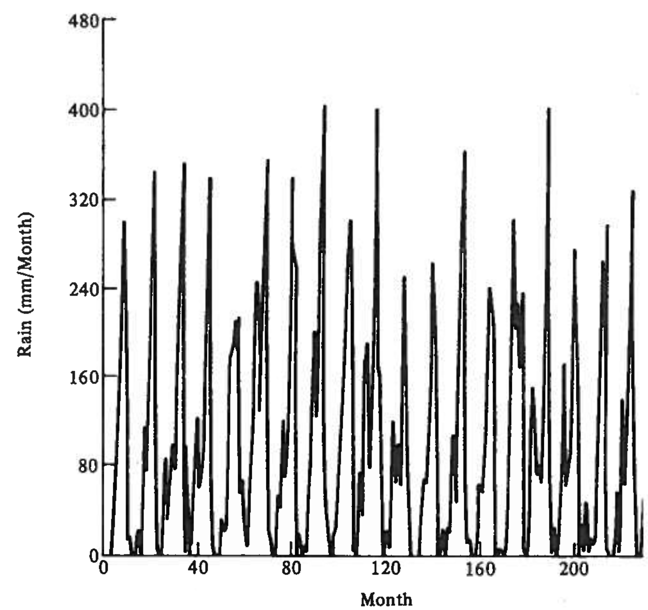

Figure 2.15 Addis Ababa Rainfall (1921–1940)

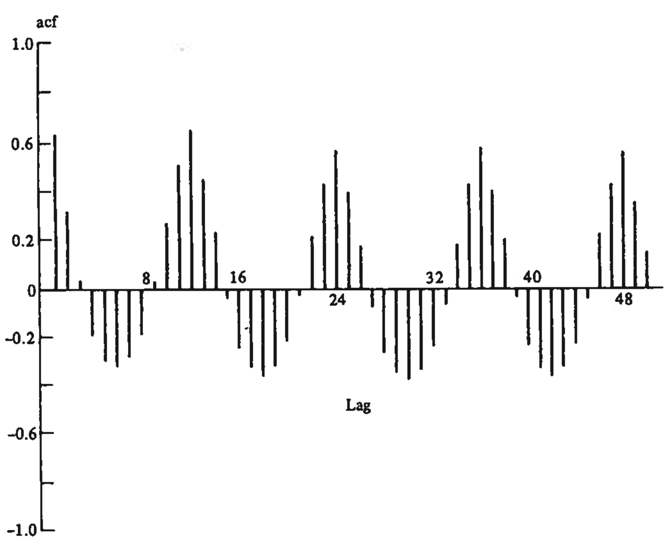

Figure 2.16 Autocorrelation function for data given for the South Saskatchewan River (from Hipel, 1975).

general ARIMA(p, d, q) model. The concept of first-order seasonal differences can be extended to higher-order, D, differences. It is further possible to suggest a seasonal ARMA structure for the ε_t on top of the nonseasonal ARIMA(p, d, q) structure. In summary the general seasonal multiplicative ARIMA model becomes

$$\phi(B)\phi(B^s)(1-B)^d(1-B^s)^D X_t = \theta(B)\theta(B^s)a_t,$$

or

$$\phi(B)\phi(B^s)\nabla^d\nabla_s^D X_t = \theta(B)\theta(B^s)a_t, \qquad (2.100)$$

where $\phi(B)$ are $\theta(B)$ are polynomials of B of degrees p and q, respectively, and $\phi(B^s)$ and $\theta(B^s)$ are polynomials of B^s of degrees P and Q, respectively. The basic seasonal period is s, d is the order of local differentiation, and D is the order of seasonal differentiation. The model in Eq. (2.100) is called an ARIMA(p, d, q)×(P, D, Q) model. Note that Eq. (2.100) represents an ARMA model (with seasonal components) operating on a stationary series, resulting

from differences of the original X_t sequence,

$$Z(t) = \nabla^d \nabla_s^D X_t. \qquad (2$$

Traditionally, hydrologists have handled seasonal nonstationarities i mean as well as variance by externally estimating the first two moment then normalizing to a zero mean, unit-variance process. Ideally, give continuous process $X(t)$, normalization yields,

$$Z(t) = \frac{X(t)-\mu(t)}{\sigma(t)}. \qquad (2$$

If nonstationarity is limited to the mean and variance, then $Z(t)$ c modeled and studied as a stationary process. Generally continuous estima $\mu(t)$ and $\sigma(t)$ result from deterministic Fourier-series analysis on the o series and on discrete estimates of the variance, respectively. Note th Fourier-series analysis it is inherently assumed that the signal is pe Furthermore, discrete data must be given at equal intervals. Chapter 4 c the concept of Fourier series. The purpose of Fourier-series analysis i present context is to identify the dominant harmonics of the process, can then be interpreted as the seasonally varying elements to be taken fr original series. For example, monthly data should have a fundamental p of 12 months with the additional possibility of biannual and quarterly c nents. Many investigators (Roesner and Yevjevich, 1966; Rodriguez-I 1968; Quimpo, 1968; and Rodríguez-Iturbe and Nordin, 1968) have us above procedure to handle seasonal nonstationarity.

Another approach, particularly useful when dealing with monthly da data at larger time intervals) is simply to compute seasonal (e.g., mo quarterly) means and variances to be used in the standardizing equatio 2.102). This and the previous method requires a considerable number of of data. For example, the mean and variance of the month of January n considered constant for all months of January and must be computed several years of observations on that month.

The time series Z_t resulting from the standardization of Eq. (2.102 still result in a series with correlations between seasons of nonstat nature. That is, the correlation is not only a function of lag but may depe the absolute time or season. Thomas and Fiering (in Maass et al., 1962; Fiering and Jackson, 1971) suggest an AR(1), with time-varying coeffi that handles seasonally varying lag-one correlations. It is commonly call univariate seasonal, lag-one, autoregressive model. It takes the following

$$(X_{t,j} - m_j) = \rho_j \frac{\sigma_j}{\sigma_{j-1}}(X_{t,j-1} - m_{j-1}) + \sigma_j(1-\rho_j^2)^{1/2} W_{t,j}, \qquad ($$

where $X_{t,j}$ is the value of random process (i.e., streamflow) at year season j,

ρ_j is the lag-one correlation coefficient between seasons j and $j-1$,
m_j is the mean value of season j,
σ_j is the standard deviation at season j, and
$W_{t,j}$ is an independent normal random variable with mean 0 and variance 1.

The above model will preserve not only the seasonal means, but also the seasonal variances and correlations of the process. To complete the definition, it must be stated that

$$X_{t,j^*+1} = X_{t+1,1}, \tag{2.104}$$

where j^* is the last season of year t.

By taking expected values of Eq. (2.103), it is obvious that a zero-mean process is defined, implying that the correct mean values are preserved. Redefining this zero-mean process as

$$Z_{t,j} = X_{t,j} - m_j,$$

the variance is

$$\begin{aligned} E\left[Z_{t,j}^2\right] &= \rho_j^2 \frac{\sigma_j^2}{\sigma_{j-1}^2} E\left[Z_{t,j-1}^2\right] + \sigma_j^2\left(1-\rho_j^2\right)E\left[W_{t,j}^2\right] \\ &= \rho_j^2 \frac{\sigma_j^2}{\sigma_{j-1}^2}\sigma_{j-1}^2 + \sigma_j^2\left(1-\rho_j^2\right) \\ &= \sigma_j^2, \end{aligned}$$

which proves preservation of each σ_j^2 assuming that σ_{j-1}^2 is preserved.

The above uses a circular argument. Assuming a two-season model, the proof of the preservation of the variance is

$$\begin{aligned} E\left[Z_{t,2}^2\right] &= \rho_2^2 \frac{\sigma_2^2}{\sigma_1^2} E\left[Z_{t,1}^2\right] + \sigma_2^2\left(1-\rho_2^2\right) \\ &= \rho_2^2 \frac{\sigma_2^2}{\sigma_1^2}\left[\rho_1^2 \frac{\sigma_1^2}{\sigma_2^2} E\left[Z_{t-1,2}^2\right] + \sigma_1^2\left(1-\rho_1^2\right)\right] + \sigma_2^2\left(1-\rho_2^2\right). \end{aligned}$$

Using the stationarity assumption,

$$\begin{aligned} E\left[Z_{t,2}^2\right] &= \rho_2^2\rho_1^2 \frac{\sigma_2^2}{\sigma_1^2}\frac{\sigma_1^2}{\sigma_2^2} E\left[Z_{t,2}^2\right] + \rho_2^2\frac{\sigma_2^2}{\sigma_1^2}\sigma_1^2\left(1-\rho_1^2\right) + \sigma_2^2\left(1-\rho_2^2\right) \\ &= \rho_2^2\rho_1^2 E\left[Z_{t,2}^2\right] + \rho_2^2\sigma_2^2\left(1-\rho_1^2\right) + \sigma_2^2\left(1-\rho_2^2\right) \\ &= \rho_2^2\rho_1^2 E\left[Z_{t,2}^2\right] + \sigma_2^2\left(1-\rho_2^2\rho_1^2\right) \end{aligned}$$

$$E\left[Z_{t,2}^2\right] = \frac{\sigma_2^2\left(1-\rho_2^2\rho_1^2\right)}{1-\rho_2^2\rho_1^2} = \sigma_2^2.$$

The lag-one covariance of the seasonal model is

$$\begin{aligned} E\left[Z_{t,j}Z_{t,j-1}\right] &= \rho_j \frac{\sigma_j}{\sigma_{j-1}} E\left[Z_{t,j-1}\right]^2 + \sigma_j\left(1-\rho_j^2\right)^{1/2} E\left[W_{t,j}Z_{t,j-1}\right] \\ &= \rho_j \frac{\sigma_j}{\sigma_{j-1}}\sigma_{j-1}^2 = \rho_j\sigma_j\sigma_{j-1}. \end{aligned}$$

The above in fact is the covariance between seasons j and $j-1$. definition, this leads to

$$\rho_j = \frac{E\left[X_{t,j}X_{t,j-1}\right]}{\sigma_j\sigma_{j-1}},$$

which shows that the lag-one correlation is preserved.

Attempts to preserve skewness with the seasonal autoregressive m follow the same ideas as presented for the single season case (Eq. 2.41). use of log-normal transformations will be illustrated in Chapter 3 on seas multivariate models.

Following Fiering and Jackson (1971), if γ_j is the skewness of season is also possible to generate a process approximately gamma-distributed the correct skewness by using an equation of the form

$$\left(X_{t,j}-m_j\right) = \rho_j \frac{\sigma_j}{\sigma_{j-1}}\left(X_{t,j-1}-m_{j-1}\right) + \sigma_j\left(1-\rho_j^2\right)^{1/2}\varepsilon_{t,j} \quad (2.1$$

where $\varepsilon_{t,j}$ is an approximately gamma-distributed variate with skewness gi by

$$\gamma_{\varepsilon j} = \frac{\left[\gamma_j - \rho_{j-1}^3\gamma_{j-1}\right]}{\left(1-\rho_j^2\right)^{1.5}}. \quad (2.1$$

The variates $\varepsilon_{t,j}$ are generated using the familiar expression

$$\varepsilon_{t,j} = \frac{2}{\gamma_{\varepsilon j}}\left(1 + \frac{\gamma_{\varepsilon j}W_{t,j}}{6} - \frac{\gamma_{\varepsilon j}^2}{36}\right)^3 - \frac{2}{\gamma_{\varepsilon j}}. \quad (2.1$$

Salas and Yevjevich (1972) have extended the concept of season varying coefficients to the AR(2) and AR(3) models while Tao and Del (1976) and Delleur and Kavvas (1978) have done the same for the gen ARMA(p,q) model, which now takes the form

$$Z_{t,j} = \sum_{k=1}^{p}\phi_{k,j}Z_{t,j-k} - \sum_{k=1}^{q}\theta_{k,j}a_{t,j-k} + a_{t,j}, \quad (2.1$$

where $Z_{t,j}$ is a standardized series resulting from Eq. (2.102). The reader is referred to their work for details of model identification, parameter estimation, and model testing. See also Salas et al. (1982) for parameter estimation of such models.

Rao and Kashyap (1973, 1974) and Kashyap and Rao (1976) suggested another stochastic model for monthly streamflow that is nonstationary in mean and variance. The model structure is

$$X(k,i) = \sum_{\ell=1}^{n_1} \alpha_\ell X(k,i-\ell) + U(i) + V(k,i) + \sum_{\ell=1}^{n_2} \alpha_{n_1+\ell} V(k,i-\ell) \quad (i=0,1,\ldots,11;\ k=0,1,2,\ldots), \tag{2.109}$$

where $X(k,i)$ is discharge during the ith month of the kth year.

$$\left.\begin{aligned} X(k,i-\ell) &= X(k-1,12+i-\ell) \\ V(k,i-\ell) &= V(k-1,12+i-\ell) \end{aligned}\right\} \text{ if } i-\ell=0.$$

$$U(i) = \alpha_0 + \sum_{j=1}^{n_3} \left(\alpha_{n_1+n_2+2j-1}\cos w_j i + \alpha_{n_1+n_2+2j}\sin w_j i\right)$$

$$w_j = 2\pi j/12$$

$$V(k,i) = \psi(i) W(k,i)$$

$$\psi(i) = \beta_0 + \sum_{j=1}^{n_4} \left(\beta_{2j-1}\cos w_j i + \beta_{2j}\sin w_j i\right)$$

$\mathbf{N} = \{n_1, n_2, n_3, n_4\}$ are structural parameters

$\boldsymbol{\alpha} = \{\alpha_0, \alpha_1, \ldots, \alpha_{n_1+n_2+2n_3}\}$ are model coefficients

$\boldsymbol{\beta} = \{\beta_0, \beta_1, \ldots, \beta_{2n_4}\}$ are model coefficients.

The random sequence $W(k,i)$ is assumed to satisfy

$$E[W(k,i)] = 0 \quad \forall k,i$$

$$E[W(k,i)W(k',i')] = \delta(k-k')\delta(i-i')\sigma_w^2 \quad \forall k,k',i,i'$$

$$E[W(k,i)X(k,i-j)] = 0 \quad \forall k,i; \qquad j>0$$

$\delta(\cdot)$ is the Kronecker delta function.

Peculiarities of the above model are the periodic deterministic function $U(i)$, which reflects the annual variation in the mean flow and the random noise of variable amplitude and variance introduced by the $V(k,i)$ term. For monthly flows the period is 12.

Identification of the order and the parameters of the above model requires considerable effort and is based on iterative procedures. The model can be used

for simulation and forecasting. Chapter 3 will discuss in more detai multivariate version of this model developed by Curry and Bras (1980).

The main difficulty of all the seasonal models previously discussed particularly those with time-varying coefficients, is their lack of parsimony number of coefficients involved can be very large and, with limited datı poorly estimated. As an illustration of this problem, the simple univ: seasonal autoregressive model has 36 parameters when used with a mo time series.

2.6 MODEL IDENTIFICATION, ESTIMATION, AND VERIFICATION

Box and Jenkins (1976) proposed an iterative algorithm for model iden tion, parameter estimation, and model verification. Figure 2.17 illustrı version of their approach.

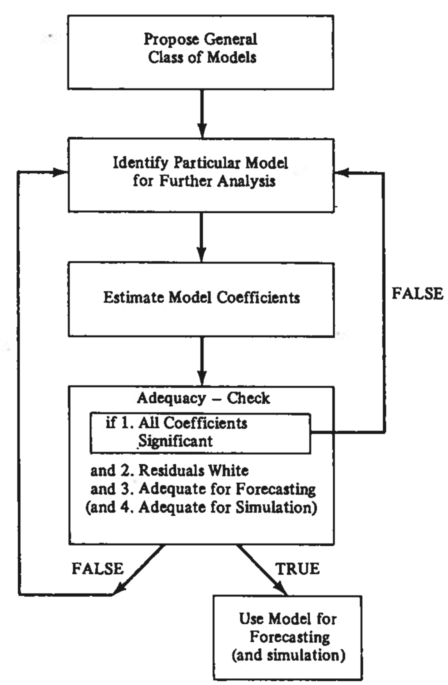

Figure 2.17 Iterative approach to model building (from Curry and Bras, 19