and $(e^{i2\pi k/n})^n = e^{i2\pi k} = 1$. Applying Equation (10.2.17) we immediately obtain the following result:

$$\sum_{t=1}^{n} e^{i2\pi kt/n} e^{-i2\pi jt/n} = \begin{cases} n, & k = j \\ 0, & k \neq j \end{cases} \tag{10.2.18}$$

That is, the system (10.2.16) is orthogonal.

10.3 FOURIER REPRESENTATION OF FINITE SEQUENCES

Let $Z_1, Z_2, \ldots, Z_n$ be a sequence of n numbers. This sequence can be regarded as a set of coordinates of a point in an n-dimensional space. In vector analysis, we often construct a set of vectors called a *basis* so that every vector in the space can be written as a linear combination of the elements of the basis. For a given n-dimensional space, it is known that any set of n orthogonal vectors forms a basis. Thus, the given sequence of n numbers, $\{Z_t\}$, can be written as a linear combination of the orthogonal trigonometric functions given in the system (10.2.3). That is,

$$Z_t = \sum_{k=0}^{[n/2]} [a_k \cos(2\pi kt/n) + b_k \sin(2\pi kt/n)], \qquad t = 1, 2, \ldots, n. \tag{10.3.1}$$

Equation (10.3.1) is called the Fourier series of the sequence Z_t. The a_k and b_k are called Fourier coefficients. Using the orthogonal property of the trigonometric functions given in (10.2.13) to (10.2.15), we obtain the a_k and b_k by multiplying $\cos(2\pi kt/n)$ and $\sin(2\pi kt/n)$ on both sides of (10.3.1), respectively, and then summing over $t = 1, 2, \ldots, n$. To avoid possible confusion, the reader can replace the index k by j in Equation (10.3.1) in this operation. Thus, we have

$$a_k = \begin{cases} \dfrac{1}{n}\displaystyle\sum_{t=1}^{n} Z_t \cos(2\pi kt/n), & k = 0, \text{ and } k = n/2 \text{ if } n \text{ is even}, \\ \dfrac{2}{n}\displaystyle\sum_{t=1}^{n} Z_t \cos(2\pi kt/n), & k = 1, 2, \ldots, \left[\dfrac{n-1}{2}\right], \end{cases} \tag{10.3.2}$$

$$b_k = \frac{2}{n}\sum_{t=1}^{n} Z_t \sin(2\pi kt/n), \qquad k = 1, 2, \ldots, \left[\frac{n-1}{2}\right].$$

Let $\omega_k = 2\pi k/n$, $k = 0, 1, \ldots, [n/2]$. These frequencies are called the Fourier frequencies. Using the system of complex exponentials given in

(10.2.16), we can also write the Fourier series of Z_t as

$$Z_t = \begin{cases} \displaystyle\sum_{k=-(n-1)/2}^{(n-1)/2} c_k e^{i\omega_k t}, & \text{if } n \text{ is odd}, \\ \displaystyle\sum_{k=-n/2+1}^{n/2} c_k e^{i\omega_k t}, & \text{if } n \text{ is even} \end{cases} \tag{10.3.3}$$

where the Fourier coefficients c_k are given by

$$c_k = \frac{1}{n}\sum_{t=1}^{n} Z_t e^{-i\omega_k t}. \tag{10.3.4}$$

Equations (10.3.1) and (10.3.3) imply that

$$\begin{aligned} Z_t &= \sum_{k=0}^{[n/2]} (a_k \cos \omega_k t + b_k \sin \omega_k t) \\ &= \begin{cases} \displaystyle\sum_{k=-(n-1)/2}^{(n-1)/2} c_k e^{i\omega_k t}, & \text{if } n \text{ is odd}, \\ \displaystyle\sum_{k=-(n/2)+1}^{n/2} c_k e^{i\omega_k t}, & \text{if } n \text{ is even}. \end{cases} \end{aligned}$$

Thus, from the relationships (10.2.5) and (10.2.6), the Fourier coefficients a b_k, and c_k are easily seen to be related as

$$\begin{cases} c_0 = a_0;\ c_{n/2} = a_{n/2} \ (\text{even } n) \\ c_k = \dfrac{a_k - ib_k}{2} \\ c_{-k} = c_k^* = \dfrac{a_k + ib_k}{2}. \end{cases} \tag{10.3.5}$$

The coefficient $c_0 = a_0 = \sum_{t=1}^{n} Z_t/n$ is often referred to as the d.c. value, i.e the constant average value of the sequence.

The above material shows that any finite sequence can be written as a lin ear combination of the sine-cosine sequences or the complex exponentials.

10.4 FOURIER REPRESENTATION OF PERIODIC SEQUENCES

A general function $f(t)$ is said to be periodic with period P if there exists positive constant P such that

$$f(t + P) = f(t) \tag{10.4.1}$$

for all t. It is obvious that a sequence that is periodic with period P is also periodic with periods $2P$, $3P$, ... The smallest positive value of P for which (10.4.1) holds is called the fundamental period or simply the period of the function.

Suppose that the sequence (or the discrete time function) Z_t is periodic with period n, where n is a positive integer. That is,

$$Z_{t+n} = Z_t \tag{10.4.2}$$

for all integer t. A fundamental phenomenon of a periodic function is that the function is uniquely determined by its pattern in a range of one period. Outside that range the function is just a repetition of the pattern in that range. Thus, a periodic sequence with period n is uniquely defined by its values at $t = 1$, 2, ..., and n.

Following the result of the Fourier series of a finite sequence in Section 10.3, we can write Z_t for $t = 1, 2, \ldots, n$ as the linear combination of the orthogonal sine and cosine functions as follows:

$$Z_t = \sum_{k=0}^{[n/2]} (a_k \cos\omega_k t + b_k \sin\omega_k t) \tag{10.4.3}$$

$$a_k = \begin{cases} \dfrac{1}{n}\displaystyle\sum_{t=1}^{n} Z_t \cos\omega_k t, & k = 0, \text{ and } k = \dfrac{n}{2} \text{ if } n \text{ is even}, \\ \dfrac{2}{n}\displaystyle\sum_{t=1}^{n} Z_t \cos\omega_k t, & k = 1, 2, \ldots, \left[\dfrac{n-1}{2}\right], \end{cases} \tag{10.4.4}$$

$$b_k = \frac{2}{n}\sum_{t=1}^{n} Z_t \sin\omega_k t, \qquad k = 1, 2, \ldots, \left[\frac{n-1}{2}\right] \tag{10.4.5}$$

with $\omega_k = 2\pi k/n$. Similarly, we can also write Z_t for $t = 1, 2, \ldots, n$ as the linear combination of complex exponentials

$$Z_t = \begin{cases} \displaystyle\sum_{k=-(n-1)/2}^{(n-1)/2} c_k e^{i\omega_k t}, & \text{if } n \text{ is odd}, \\ \displaystyle\sum_{k=-(n/2)+1}^{n/2} c_k e^{i\omega_k t}, & \text{if } n \text{ is even}, \end{cases} \tag{10.4.6}$$

$$c_k = \frac{1}{n}\sum_{t=1}^{n} Z_t e^{-i\omega_k t}. \tag{10.4.7}$$

The Fourier coefficients a_k and b_k in (10.4.4) and (10.4.5), and c_k in (10.4.7) are related by the equations in (10.3.5).

It is easy to see that the sine-cosine functions $\sin\omega_k t$ and $\cos\omega_k t$ in (10.4.3) and the complex exponentials $e^{i\omega_k t}$ in (10.4.6) are periodic with period n. This

leads to the desired consequence that

$$Z_{t+jn} = Z_t,$$

for all integers t and j. In other words, the Equations (10.4.3) and (10.4.6) ar valid for all integers t.

We have demonstrated how to represent an arbitrary periodic sequenc of period n as a linear combination of n trigonometric sequences or n com plex exponentials. The smallest positive value of n for which the Fourier serie representation in (10.4.3) and (10.4.6) holds is called the fundamental period and the corresponding value $2\pi/n$ is called the fundamental frequency. Th terms for $k = +1$ and $k = -1$ in the above representation both have the sam fundamental period equal to n (and hence the same fundamental frequenc $2\pi/n$) and are collectively referred to as the first harmonic components. Mor generally, the terms for $k = +j$ and $k = -j$ both have the frequency $j(2\pi/n$ and are referred to as the jth harmonic components. Thus, all the terms in th Fourier series representation have frequencies that are multiples of the sam fundamental frequency, $2\pi/n$, and hence are harmonically related.

For a given periodic sequence Z_t of period n, the energy associated wit the sequence in one period is defined as

$$\sum_{t=1}^{n} Z_t^2. \tag{10.4.8}$$

Now, multiplying Z_t on the both sides of (10.4.3), summing from $t = 1$ to $t = n$ and using the relation (10.4.4) and (10.4.5), we have

$$\sum_{t=1}^{n} Z_t^2 = \sum_{k=0}^{[n/2]} \left[a_k \sum_{t=1}^{n} Z_t \cos\omega_k t + b_k \sum_{t=1}^{n} Z_t \sin\omega_k t\right]$$

$$= \begin{cases} n a_0^2 + \dfrac{n}{2}\displaystyle\sum_{k=1}^{[(n-1)/2]} (a_k^2 + b_k^2), & \text{if } n \text{ is odd}, \\ n a_0^2 + \dfrac{n}{2}\displaystyle\sum_{k=1}^{[(n-1)/2]} (a_k^2 + b_k^2) + n a_{\frac{n}{2}}^2, & \text{if } n \text{ is even}. \end{cases} \tag{10.4.9}$$

Equation (10.4.9) is known as Parseval's relation for Fourier series. Equiva lently, using Equations (10.4.6) and (10.4.7), we have the corresponding forn of Parseval's relation as

$$\sum_{t=1}^{n} Z_t^2 = \begin{cases} n \displaystyle\sum_{k=-(n-1)/2}^{(n-1)/2} |c_k|^2, & \text{if } n \text{ is odd}, \\ n \displaystyle\sum_{k=-n/2+1}^{n/2} |c_k|^2, & \text{if } n \text{ is even}, \end{cases} \tag{10.4.10}$$

where $|c_k|^2 = c_k c_k^*$.

Equations (10.4.9) and (10.4.10) imply that the total energy for a periodic sequence over the whole time horizon $t = 0, +1, +2, \ldots$ is infinite. Hence, we consider its energy per unit time, which is called the power of the sequence. This is given by

$$\text{Power} = \begin{cases} a_0^2 + \dfrac{1}{2}\displaystyle\sum_{k=1}^{[(n-1)/2]} (a_k^2 + b_k^2), & \text{if } n \text{ is odd,} \\ a_0^2 + \dfrac{1}{2}\displaystyle\sum_{k=1}^{[(n-1)/2]} (a_k^2 + b_k^2) + a_{n/2}^2, & \text{if } n \text{ is even} \end{cases} \tag{10.4.11}$$

or equivalently

$$\text{Power} = \begin{cases} \displaystyle\sum_{k=-(n-1)/2}^{(n-1)/2} |c_k^2|, & \text{if } n \text{ is odd,} \\ \displaystyle\sum_{k=-n/2+1}^{n/2} |c_k^2|, & \text{if } n \text{ is even.} \end{cases} \tag{10.4.12}$$

As noted above, the jth harmonic components include the terms for both $k = +j$ and $k = -j$ as they correspond to the same frequency $j(2\pi/n)$. Hence, we can interpret the quantity

$$\begin{cases} f_0 = c_0^2 = a_0^2;\ f_{n/2} = |c_{n/2}|^2 \ (\text{even } n) \\ f_k = |c_{-k}|^2 + |c_k|^2 = 2|c_k|^2 = \dfrac{1}{2}(a_k^2 + b_k^2) \end{cases} \tag{10.4.13}$$

from the term in the Fourier series of Z_t at the kth frequency $\omega_k = 2\pi k/n$ as the contribution to the total power. The quantity f_k plotted as a function of ω_k shown in Figure 10.1 is called the power spectrum and describes how the total power is distributed over the various frequency components of the sequence Z_t.

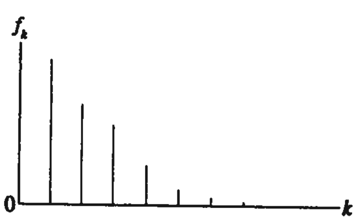

Fig. 10.1 Power spectrum.

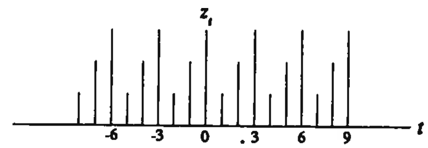

Fig. 10.2 Periodic sequence in Example 10.1.

Example 10.1 Let $Z_1 = 1$, $Z_2 = 2$, $Z_3 = 3$ and $Z_{t+3j} = Z_t$, $t = 1, 2$ $j = \pm 1, \pm 2, \pm 3, \ldots$. Clearly the function is periodic, with period 3 as shc in Figure 10.2.

Now, the period $n = 3$, which is odd and $[n/2] = 1$. The Fourier repres tation using (10.4.3) becomes

$$Z_t = a_0 + a_1 \cos(2\pi t/3) + b_1 \sin(2\pi t/3)$$

where the Fourier coefficients are obtained through (10.4.4) as

$$a_0 = \frac{1}{n}\sum Z_t = \bar{Z} = \frac{1}{3}(1+2+3) = 2$$
$$a_1 = \frac{2}{3}[1\cos(2\pi/3) + 2\cos(4\pi/3) + 3\cos(6\pi/3)] = 1$$
$$b_1 = \frac{2}{3}[1\sin(2\pi/3) + 2\sin(4\pi/3) + 3\sin(6\pi/3)] = -.5773503.$$

That is,

$$Z_t = 2 + \cos(2\pi t/3) - .5773503\sin(2\pi t/3), \qquad t = 1, 2, 3, 4, 5, \ldots.$$

Similarly, in terms of (10.4.6), we can also represent Z_t as

$$Z_t = c_{-1}e^{-i2\pi t/3} + c_0 + c_1 e^{i2\pi t/3}, \qquad t = 1, 2, 3, 4, \ldots$$

where the coefficients are calculated using (10.4.7)

$$c_0 = \frac{1}{3}[1+2+3] = 2$$
$$c_{-1} = \frac{1}{3}[e^{i2\pi/3} + 2e^{i4\pi/3} + 3e^{i6\pi/3}] = \frac{1}{2}(1 - .5773503i)$$
$$c_1 = \frac{1}{3}[e^{-i2\pi/3} + 2e^{-i4\pi/3} + 3e^{-i6\pi/3}] = \frac{1}{2}(1 + .5773503i).$$

That is,

$$Z_t = \frac{1}{2}(1 - .5773503i)e^{-i2\pi t/3} + 2 + \frac{1}{2}(1 + .5773503i)e^{i2\pi t/3},\ t = 1, 2, 3, 4, .$$

The coefficients a_k, b_k, and c_k are clearly related as shown in (10.3.5).

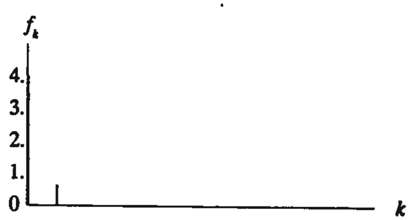

Fig. 10.3 Power spectrum of Example 10.1.

Using (10.4.13), the power spectrum of the sequence is given by

$$f_k = \begin{cases} 2^2 = 4, & k = 0, \\ \frac{1}{2}[(1)^2 + (-.5773503)^2] = \frac{2}{3}, & k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The plot is given in Figure 10.3.

10.5 FOURIER REPRESENTATION OF NONPERIODIC SEQUENCES—THE DISCRETE-TIME FOURIER TRANSFORM

Consider a general nonperiodic sequence or discrete-time function, Z_t, of finite duration such that $Z_t = 0$ for $|t| > M$. Let $n = (2M + 1)$ and define a new function

$$Y_{t+jn} = Z_t, \; -(n-1)/2 \le t \le (n-1)/2, \qquad j = 0, \pm 1, \pm 2, \ldots \tag{10.5.1}$$

which is clearly periodic with period n. Thus, using (10.4.6), we can express (10.5.1) in a Fourier series of the form

$$Y_t = \sum_{k=-(n-1)/2}^{(n-1)/2} c_k e^{i2\pi kt/n}, \tag{10.5.2}$$

$$c_k = \frac{1}{n} \sum_{t=-(n-1)/2}^{(n-1)/2} Y_t e^{-i2\pi kt/n}. \tag{10.5.3}$$

However, in the interval $-(n-1)/2 \le t \le (n-1)/2$, $Y_t = Z_t$. Therefore,

$$c_k = \frac{1}{n} \sum_{t=-(n-1)/2}^{(n-1)/2} Z_t e^{-i2\pi kt/n}$$

$$= \frac{1}{n} \sum_{t=-\infty}^{\infty} Z_t e^{-i2\pi kt/n} \tag{10.5.4}$$

where we have used the fact that $Z_t = 0$ for $t < -(n-1)/2$ or $t > (n-1)/2$. Let

$$f(\omega) = \frac{1}{2\pi} \sum_{t=-\infty}^{\infty} Z_t e^{-i\omega t}. \tag{10.5.5}$$

We have

$$c_k = \frac{2\pi}{n} f(k\Delta\omega) \tag{10.5.6}$$

where $\Delta\omega = 2\pi/n$ is the sample spacing of frequencies. Combining (10.5.2) and (10.5.6) yields

$$Y_t = \sum_{k=-(n-1)/2}^{(n-1)/2} \frac{2\pi}{n} f(k\Delta\omega) e^{ik\Delta\omega t}. \tag{10.5.7}$$

Since $\Delta\omega = 2\pi/n$, (10.5.7) can be written as

$$Y_t = \sum_{k=-(n-1)/2}^{(n-1)/2} f(k\Delta\omega) e^{ik\Delta\omega t} \Delta\omega. \tag{10.5.8}$$

Now each term in the summation of (10.5.8) represents the area of a rectangle of height $f(k\Delta\omega)e^{ik\Delta\omega t}$ and width $\Delta\omega$. As $n \to \infty$, we have $Y_t \to Z_t$ and $\Delta\omega \to 0$. Thus, the limit of the summation becomes an integral. Furthermore, because the summation is carried out over n consecutive values of width $\Delta\omega = 2\pi/n$, the total interval of integration will always have a width of 2π. Therefore,

$$\begin{aligned} Z_t &= \lim_{n\to\infty} Y_t \\ &= \lim_{\Delta\omega\to 0} \sum_{k=-\infty}^{\infty} f(k\Delta\omega) e^{ik\Delta\omega t} \Delta\omega \\ &= \int_{2\pi} f(\omega) e^{i\omega t}\, d\omega. \end{aligned} \tag{10.5.9}$$

Since $f(\omega)e^{i\omega t}$, as a function of ω, is periodic with period 2π, the interval of integration can be taken as any interval of length 2π. Specifically, we can consider $-\pi \le \omega \le \pi$. Thus, we have the following:

$$Z_t = \int_{-\pi}^{\pi} f(\omega) e^{i\omega t}\, d\omega, \qquad t = 0, \pm 1, \pm 2, \ldots \tag{10.5.10}$$

and

$$f(\omega) = \frac{1}{2\pi} \sum_{t=-\infty}^{\infty} Z_t e^{-i\omega t}, \qquad -\pi \le \omega \le \pi. \tag{10.5.11}$$

The function $f(\omega)$ in (10.5.11) is usually referred to as the (discrete time) Fourier transform of Z_t, and Z_t in (10.5.10) is often referred to as the (discrete time) inverse Fourier transform of $f(\omega)$. They form a Fourier transform pair.

From the above discussion, we see that we can also define the Fourier transform $f(\omega)$ as

$$f(\omega) = \sum_{t=-\infty}^{\infty} Z_t e^{-i\omega t} \tag{10.5.12}$$

or

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \sum_{t=-\infty}^{\infty} Z_t e^{-i\omega t} \tag{10.5.13}$$

instead of the given (10.5.5). These modified definitions of $f(\omega)$ lead to the following Fourier transform pairs:

$$\begin{cases} Z_t = & \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(\omega) e^{i\omega t}\, d\omega \\ f(\omega) = & \displaystyle\sum_{t=-\infty}^{\infty} Z_t e^{-i\omega t} \end{cases} \tag{10.5.14}$$

and

$$\begin{cases} Z_t = & \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\pi}^{\pi} f(\omega) e^{i\omega t}\, d\omega \\ f(\omega) = & \dfrac{1}{\sqrt{2\pi}} \displaystyle\sum_{t=-\infty}^{\infty} Z_t e^{-i\omega t} \end{cases} \tag{10.5.15}$$

respectively, which may be found in other books. In this book, we use the Fourier transform pairs as given in (10.5.10) and (10.5.11).

The derivation of the integral in (10.5.10) as the limiting form of a sum implies that the inverse Fourier transform given in (10.5.10) represents the sequence Z_t as a linear combination of complex sinusoids infinitesimally close in frequency with amplitudes $|f(\omega)|(d\omega)$. Thus, the quantity $|f(\omega)|$ is often referred to as the spectrum or amplitude spectrum of the sequence, as it provides us with the information on how Z_t is composed of complex sinusoids at different frequencies.

The energy associated with the sequence is given by the following Parseval's relation:

$$\sum_{t=-\infty}^{\infty} |Z_t|^2 = \sum_{t=-\infty}^{\infty} Z_t \int_{-\pi}^{\pi} f^*(\omega) e^{-i\omega t}\, d\omega$$

$$= \int_{-\pi}^{\pi} f^*(\omega) \sum_{t=-\infty}^{\infty} Z_t e^{-i\omega t}\, d\omega$$

$$= 2\pi \int_{-\pi}^{\pi} f^*(\omega) f(\omega)\, d\omega$$

$$= 2\pi \int_{-\pi}^{\pi} |f(\omega)|^2\, d\omega. \qquad \text{(10.:}$$

Parseval's relation thus relates energy in the time domain to energy i frequency domain. In other words, the energy may be determined eith computing the energy per unit time $|Z_t|^2$ and summing over all time, computing the energy per unit frequency $2\pi|f(\omega)|^2$ and integrating ov frequencies. Hence, $g(\omega) = 2\pi|f(\omega)|^2$, as a function of ω, is also referred the energy spectrum or the energy spectral density function.

In the above construction, Z_t was assumed to be of arbitrary finite tion. Equations (10.5.10) and (10.5.11) remain valid for a general nonper sequence of infinite duration. However, in this case, we must consider the tion of convergence of the infinite summation in (10.5.11). A condition which guarantees the convergence of this sum is that the sequence $\{Z_t\}$ i solutely summable. That is,

$$\sum_{t=-\infty}^{\infty} |Z_t| < \infty. \qquad \text{(10.:}$$

In fact, the above theory holds also when Z_t is square summable, i.e.,

$$\sum_{t=-\infty}^{\infty} Z_t^2 < \infty. \qquad \text{(10.:}$$

The proof of the result using a weak condition depends on an alternative sentation of the Fourier transform and is omitted. For our purposes, it su to note that the condition in Equation (10.5.17) implies the condition in E tion (10.5.18), but the converse is not true.

It is of interest to note the important differences between the frequ domain properties of periodic and nonperiodic sequences summarized i following remarks.

1. The spectrum frequencies of periodic sequences are harmonically re and form a finite discrete set, whereas those of nonperiodic sequences a continuum of frequencies.
2. The energy over whole time horizon $t = 0, \pm 1, \pm 2, \ldots$ for periodi quences is infinite. Hence, we study their properties in terms of the p spectrum over a finite set of harmonically related frequencies. The r ing spectra are hence sometimes referred to as line spectra. The energy the whole time horizon for nonperiodic sequences is finite as guarante the condition in Equation (10.5.17). Hence, we describe their propert terms of the energy spectrum over a continuum of frequencies.