Figure 2.38   Predictions of model 2 for the Orinoco River at Palua starting August 9, 1977. (A. Sancholuz, S. Carrasquel, D. Vargas, Laboratorio Nacional de Hidráulica, 1981. Under contract by INC, Caracas, Venezuela.)

(60 days) forecasts in the Orinoco over several periods. Unfortunately, the examples include periods used during calibration. Behavior over only two periods are shown here, all involving predictions of the rising limb and peak of the hydrograph. Figures 2.35 and 2.36 show 12-lead predictions of models 1 and 2, respectively, for the period starting June 10, 1977. Similarly Figs. 2.37 and 2.38 are predictions for the period starting August 9, 1977. That is, the figures show two consecutive prediction periods of 60 days each. Both models performed relatively well; error magnitudes are also given in the figures. The model developers concluded that of all tests, model 1 had a 50% probability of *making an error of less than ± 0.47 m at the one-month (lead-6) forecast and of ± 0.57 m at the two-month (lead-12) forecast.* Model 1, though, required standardizing the series by first estimating 73 daily flow means and 73 corresponding variances. Lead-6 predictions with model 2 were in error by less than ± 0.49 m 50% of the time. The equivalent lead-12 error statistic was ± 0.61 m. Nevertheless, no standardization or estimation of means and variances were required by this more parsimonious model.

CHAPTER 3

Multivariate Time-Series Analysis

3.1   INTRODUCTION

Hydrologic studies are commonly multivariate in nature. For example, river-basin planning usually involves development of multiple sites, all of which are naturally related. Reservoir operation cannot be independent of other impoundments in the same river network. Rainfall is usually sampled at discrete, related locations. Therefore it is necessary to consider jointly data from the various rain gages.

Although conceptually multivariate time-series analysis follows the same ideas presented in Chapter 2, in practice the mathematics and the theory lag behind. This lag in development responds to computational and theoretical difficulties. Of the latter, most important is the lack of a unified approach to represent jointly each of the random processes described by a different stochastic-process model. The most serious computational difficulty is parameter estimation. The quantities of data required for adequate parameter estimation in a multivariate model can be unmanageable. The mathematics of sophisticated estimation procedures also become burdensome.

Multivariate stochastic hydrologic modeling mostly has followed the philosophy of fitting limited moments of historical time series. This is in contrast to the philosophy of extensive identification, estimation, and verification of Chapter 2, which emphasized more detailed reproduction of the properties of the original time series. This chapter will first discuss multivariate models of the ARMA type. Next, seasonal multivariate models will be discussed. Among these, a general ARMA-type seasonal model will be developed in detail, following the necessary steps of identification, estimation, and verification. Finally, as an alternative to seasonal simulation, the popular disaggregation model will be presented.

3.2 MULTIVARIATE STATIONARY MODELS

3.2.1 Multivariate AR(1)

By far the most popular multivariate hydrologic streamflow model is the autoregressive lag-one. Its use in hydrology was first suggested by Matalas (1967). Its simplest form is

$$\mathbf{Z}(t) = \mathbf{A}\mathbf{Z}(t-1) + \mathbf{B}\boldsymbol{\varepsilon}(t),\qquad (3.1)$$

where

$$\mathbf{Z}^T(t) = [Z_1(t)\ldots Z_n(t)]$$
$$\boldsymbol{\varepsilon}^T(t) = [\varepsilon_1(t)\ldots\varepsilon_n(t)]$$

and $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ parameter matrices. The vector $\mathbf{Z}(t)$ is composed of n different but interdependent, zero-mean time series. For example, they may be runoff or rainfall at n different locations. They could also be related series of rainfall and runoff. The vector, $\boldsymbol{\varepsilon}(t)$ consists of n uncorrelated shocks (white noise) of zero mean and unit variance. The vector covariance of $\boldsymbol{\varepsilon}(t)$ is the identity matrix,

$$E[\boldsymbol{\varepsilon}(t)\boldsymbol{\varepsilon}^T(t)] = \mathbf{I}.$$

Vector $\boldsymbol{\varepsilon}(t)$ is uncorrelated with $\mathbf{Z}(\tau)$, for $\tau < t$:

$$E[\mathbf{Z}(\tau)\boldsymbol{\varepsilon}^T(t)] = 0.$$

Note that following the convention of Chapter 2, $\mathbf{Z}(t)$ is a zero-mean vector. Generally it would be possible to state that

$$\mathbf{X}(t) = \mathbf{Z}(t) + \mathbf{m},\qquad (3.2)$$

where $\mathbf{m}$ is an $n \times 1$ vector of constant means of the original process $\mathbf{X}(t)$.

The parameter matrices $\mathbf{A}$ and $\mathbf{B}$ are generally obtained using the method of moments. Therefore they are estimated in order to preserve explicitly a limited number of moments of the original time series. Analogous to the Thomas–Fiering model described in Chapter 2 the preserved moments are the covariance and lag-one covariance of the process.

Equation (3.1) can be post-multiplied by the transpose of $\mathbf{Z}(t)$ and expectations taken. This amounts to computing the covariance of $\mathbf{Z}(t)$,

$$E[\mathbf{Z}(t)\mathbf{Z}^T(t)] = \mathbf{A}E[\mathbf{Z}(t-1)\mathbf{Z}^T(t)]$$
$$+ \mathbf{B}E[\boldsymbol{\varepsilon}(t)[\mathbf{Z}^T(t-1)\mathbf{A}^T + \boldsymbol{\varepsilon}^T(t)\mathbf{B}^T]].\qquad (3.3)$$

Defining the covariance

$$\mathbf{M}_0 = E[\mathbf{Z}(t)\mathbf{Z}^T(t)]$$

and the covariance at lag one

$$\mathbf{M}_1^T = E[\mathbf{Z}(t-1)\mathbf{Z}^T(t)]$$

and exploiting the white-noise properties of $\boldsymbol{\varepsilon}(t)$, reduces Eq. (3.3) to

$$\mathbf{M}_0 = \mathbf{A}\mathbf{M}_1^T + \mathbf{B}\mathbf{B}^T.\qquad (3.4)$$

Equation (3.4) relates moments $\mathbf{M}_0$ and $\mathbf{M}_1$ to parameters $\mathbf{A}$ and $\mathbf{B}$ through a quadratic matrix equation. In order to specify the system completely another equation is required.

Equation (3.1) can now be post-multiplied by $\mathbf{Z}^T(t-1)$ and expected values taken to define the lag-one covariance,

$$E[\mathbf{Z}(t)\mathbf{Z}^T(t-1)] = \mathbf{A}E[\mathbf{Z}(t-1)\mathbf{Z}^T(t-1)] + \mathbf{B}E[\boldsymbol{\varepsilon}(t)\mathbf{Z}^T(t-1)].$$

Again using the white-noise properties of $\boldsymbol{\varepsilon}(t)$, the above reduces to

$$\mathbf{M}_1 = \mathbf{A}\mathbf{M}_0,\qquad (3.5)$$

where system stationarity was invoked to state

$$E[\mathbf{Z}(t)\mathbf{Z}^T(t)] = E[\mathbf{Z}(t-1)\mathbf{Z}^T(t-1)] = \mathbf{M}_0.\qquad (3.6)$$

Equation (3.5) can be solved for $\mathbf{A}$ if $\mathbf{M}_0$ is a valid covariance matrix, i.e., it is positive definite.

$$\mathbf{A} = \mathbf{M}_1\mathbf{M}_0^{-1}.\qquad (3.7)$$

Given Eq. (3.7), Eq. (3.4) reduces to a quadratic expression for $\mathbf{B}$,

$$\mathbf{B}\mathbf{B}^T = \mathbf{M}_0 - \mathbf{M}_1\mathbf{M}_0^{-1}\mathbf{M}_1^T.\qquad (3.8)$$

The form of Eq. (3.8) will recur throughout the formulation of multivariate ARMA models. There are several possible solutions to $\mathbf{B}$, given its quadratic form $\mathbf{BB}^T$. Two of these solutions are discussed in the next subsection and their use will be required whenever equations of the form of Eq. (3.8) appear.

Assuming, as the next subsection will prove, that Eq. (3.8) has a solution, then expressions to obtain $\mathbf{A}$ and $\mathbf{B}$ will be available. Note that the above model collapses to the Thomas–Fiering model (AR(1)) when only one time series is involved. In that case, letting $n = 1$,

$$\mathbf{M}_0 = \sigma_z^2$$
$$\mathbf{M}_1 = \text{cov}[Z_t Z_{t-1}]$$

and

$$\rho_1 = \frac{\mathbf{M}_1}{\mathbf{M}_0}.$$

Therefore

$$\mathbf{A} = \frac{\mathbf{M}_1}{\mathbf{M}_0} = \rho_1$$

and

$$\mathbf{BB}^T = \sigma_z^2 - \sigma_z^2 \rho_1^2$$
$$= \sigma_z^2 (1 - \rho_1^2)$$

or

$$\mathbf{B} = (1 - \rho_1^2)^{1/2} \sigma_z.$$

The above expressions for $\mathbf{A}$ and $\mathbf{B}$ correspond to those obtained in Chapter 2.

Note that a typical element of vector $\mathbf{Z}(t)$ is generated from an equation of the form

$$Z_i(t) = \sum_{j=1}^n a_{ij} Z_j(t-1) + \sum_{j=1}^n b_{ij} \varepsilon_j(t) \qquad \forall_i, \qquad (3.9)$$

which is similar to a regression equation with correlated residuals. In fact the elements of the matrix $\mathbf{A}_{ij}$, a_{ij}, are generalized least-square regression coefficients.

Operationally, estimates of $\mathbf{A}$ and $\mathbf{B}$ are obtained from Eqs. (3.7) and (3.8) by using sample estimates $\hat{\mathbf{M}}_0$ and $\hat{\mathbf{M}}_1$ of the covariance and lag-one covari-

ance matrices, respectively. These estimates have the forms:

$$\hat{\mathbf{M}}_0 = \begin{bmatrix} S_{z_1}^2 & \cdots & r_{z_1 z_n}^0 S_{z_1} S_{z_n} \\ \vdots & \ddots & \vdots \\ r_{z_1 z_n}^0 S_{z_1} S_{z_n} & \cdots & S_{z_n}^2 \end{bmatrix} \qquad (3.10)$$

$$\hat{\mathbf{M}}_1 = \begin{bmatrix} r_{z_1 z_1}^1 S_{z_1}^2 & \cdots & r_{z_1 z_n}^1 S_{z_1} S_{z_n} \\ \vdots & \ddots & \vdots \\ r_{z_n z_1}^1 S_{z_1} S_{z_n} & \cdots & r_{z_n z_n}^1 S_{z_n}^2 \end{bmatrix}, \qquad (3.11)$$

where S_{z_i} is the sample standard deviation of variable Z_i, and $r_{z_i z_j}^k$ is the correlation between variables Z_i and Z_j at lag k.

The elements of the above matrices can be computed using traditional estimators. As discussed in the subsection below, numerical stability and consistency require some care in defining sample statistics. The following estimators are recommended.

$\overline{\mathbf{X}}$ is the matrix of historical (observed) values of the vector $\mathbf{Z}(t)$, each column is an observed year of $\mathbf{Z}(t)$. If there are N observed years, then $\overline{\mathbf{X}}$ is an $n \times N$ matrix. $\overline{\mathbf{Y}}$ is the matrix of historical (observed) values of the vector $\mathbf{Z}(t-1)$, each column being an observed year of $\mathbf{Z}(t-1)$. Given N years of observations $\overline{\mathbf{Y}}$ is $n \times N$. Note that in this case $\overline{\mathbf{Y}}$ is the same as $\overline{\mathbf{X}}$ lagged by one year. Therefore, $N+1$ years of data are actually required. The sample estimates of $\mathbf{M}_0$ and $\mathbf{M}_1$ are then,

$$\hat{\mathbf{M}}_0 = \frac{1}{N} \overline{\mathbf{X}} \overline{\mathbf{X}}^T \qquad (3.12)$$

$$\hat{\mathbf{M}}_1 = \frac{1}{N} \overline{\mathbf{X}} \overline{\mathbf{Y}}^T. \qquad (3.13)$$

Finally, it is useful to point out that $\mathbf{M}_0$ is a symmetrical matrix, but $\mathbf{M}_1$ is not.

Decomposition of $\mathbf{BB}^T$

Matrices of the form $\mathbf{D} = \mathbf{BB}^T$ are called Gramian matrices, $\mathbf{D}$ being the Gramian of $\mathbf{B}$. It can be shown that $\mathbf{D}$ must be positive or positive-semidefinite, and in fact that any positive or positive-semidefinite matrix is a Gramian of some matrix (Valencia and Schaake, 1972). Given $\mathbf{D}$, there are infinite solutions for $\mathbf{B}$, since the above equation is satisfied by any matrix of the form $\mathbf{B} \cdot \mathbf{C}$, where $\mathbf{C}$ is orthogonal, implying $\mathbf{CC}^T = \mathbf{I}$. For any such $\mathbf{C}$,

$$\mathbf{D} = \mathbf{BCC}^T \mathbf{B}^T = \mathbf{BB}^T. \qquad (3.14)$$

There are two popular solutions for **B** in Eq. (3.14). The first uses the properties of principal components and is able to handle the case when **D** is positive semidefinite, implying that **B** has rank less than n, if $n \times n$ are the dimensions of **D**. This is useful in dealing with a particular class of multivariate models—the Disaggregation Model—which will be discussed in Section 3.5.

The procedure uses the properties of eigenvalues and eigenvectors. Call,

$$\mathbf{D} = \mathbf{B}\mathbf{B}^T. \tag{3.15}$$

D is always a positive-semidefinite matrix.

Define a matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & \vdots & \cdots & \vdots & \mathbf{P}_n \end{bmatrix},$$

where $\mathbf{P}_i = i$th eigenvector of matrix **D**.

The matrix **P** has the property of orthogonality.

$$\mathbf{P}^T\mathbf{P} = \mathbf{P}\mathbf{P}^T = \mathbf{I}.$$

Define $e_1, \ldots, e_n$, the eigenvalues of matrix **D**, and using the properties of eigenvalues and eigenvectors,

$$\mathbf{D}\mathbf{P}_i = \mathbf{P}_i e_i. \tag{3.16}$$

Defining a diagonal matrix of eigenvalues,

$$\mathbf{E} = \begin{bmatrix} e_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & e_n \end{bmatrix},$$

the system of equations given by Eq. (3.16) can be represented as

$$\mathbf{D}\mathbf{P} = \mathbf{P}\mathbf{E} \tag{3.17}$$

or

$$\mathbf{D} = \mathbf{P}\mathbf{E}\mathbf{P}^{-1} = \mathbf{B}\mathbf{B}^T. \tag{3.18}$$

Therefore,

$$\mathbf{B} = \mathbf{P}\mathbf{E}^{1/2}, \tag{3.19}$$

where

$$\mathbf{E}^{1/2} = \begin{bmatrix} e_1^{1/2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e_n^{1/2} \end{bmatrix}.$$

The **B** matrix obtained in this manner is of rank $n - \ell$, where ℓ is the number of zero eigenvalues in **D**.

Numerically, the above procedure is limited by the algorithms used in finding eigenvalues and eigenvectors. For large matrices such procedures may result in errors and instabilities.

A second procedure, valid for positive-definite matrices, suggests a lower triangular form for matrix **B**,

$$\mathbf{B} = \begin{bmatrix} b_{11} & 0 & \cdot & \cdots & 0 \\ b_{21} & b_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ b_{n1} & \cdot & \cdot & \cdots & b_{nn} \end{bmatrix}.$$

Multiplying **B** by its transpose,

$$\mathbf{D} = \mathbf{B}\mathbf{B}^T$$

$$= \begin{bmatrix} b_{11}^2 & b_{11}b_{21} & b_{11}b_{31} & \cdots & b_{11}b_{n1} \\ b_{21}b_{11} & b_{21}^2 + b_{22}^2 & b_{21}b_{31} + b_{22}b_{32} & \cdots & b_{21}b_{n1} + b_{22}b_{n2} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n1}b_{11} & b_{n1}b_{21} + b_{22}b_{n2} & \cdots & \cdots & \sum_{i-1}^{n} b_{nn}^2 \end{bmatrix}.$$

From the above matrix equation, the elements of matrix **B** can be obtained through sequential algebraic equations. For example, defining d_{ij} as an element of **D**,

$$\begin{aligned} b_{11} &= \sqrt{d_{11}} \\ b_{21} &= d_{12}/\sqrt{d_{11}} \\ &\vdots \\ b_{n1} &= d_{1n}/\sqrt{d_{11}}. \end{aligned} \tag{3.20}$$

In general multiplying the ith row of **B** by the ith column of $\mathbf{B}^T$ yields

$$b_{i1}^2 + b_{i2}^2 + \cdots + b_{ii}^2 = d_{ii}$$

on solving for b_{ii}:

$$b_{ii} = \left(d_{ii} - \left[b_{i1}^2 + \cdots + b_{ii-1}^2 \right] \right)^{1/2}, \tag{3.21}$$

assuming the quantity in parenthesis is positive, which is the positive-definite condition.

Given the first element of column i, b_{ii}, the remaining parameters result from

$$b_{ji} = \frac{d_{ij} - \left[b_{i1}b_{j1} + b_{i2}b_{j2} + \cdots + b_{ii-1}b_{ji-1} \right]}{b_{ii}}, \tag{3.22}$$

with $j > i$ and $b_{ii} \neq 0$.

The above procedure was first suggested for use in hydrology by Young and Pisano (1968). It is computationally fast and accurate for small matrices. When dealing with large matrices, numerical accuracy deteriorates due to the many sequential multiplications and additions required. In such cases the first method is as fast as and more accurate than the second, assuming that a reasonable algorithm to obtain eigenvalues and eigenvectors is used. Note that the first method is computationally feasible even if zero or negative eigenvalues result from numerical errors. The second method will fail any time any b_{ii} is zero or if the term in parenthesis in Eq. (3.21) is negative, a possible numerical anomaly in practice.

In multivariate AR(1), $\mathbf{BB}^T$ (Eq. 3.8) is a covariance matrix; in fact it is the conditional covariance of $\mathbf{Z}(t)$ given past observations $\mathbf{Z}(t-1)$. As such, $\mathbf{BB}^T$ must be positive-definite, assuming we are dealing with stationary processes. This will be the case any time matrices of the $\mathbf{BB}^T$ form appear in the definition of multivariate ARMA models.

Values of $\mathbf{BB}^T$ resulting from sample statistics, i.e., $\hat{\mathbf{M}}_0$ and $\hat{\mathbf{M}}_1$, can nevertheless in fact be nonpositive-definite. Such unexpected results are either of statistical origin or related to numerical accuracy of computational algorithms. They will commonly occur if the elements of $\hat{\mathbf{M}}_0$ and $\hat{\mathbf{M}}_1$ are calculated from data sets of different lengths; this yields sample statistics of differing accuracy and variability. A useful reference in this case is the paper of Crosby and Maddock (1970). The use of different statistical procedures in defining the elements of the necessary covariance matrices can also result in inconsistent $\mathbf{BB}^T$. To avoid such difficulties the estimators given in Eqs. (3.12) and (3.13) are recommended.

It is still possible to obtain inconsistent $\mathbf{BB}^T$ matrices. Such failures can be attributed to data transformations, as will be discussed later, or more commonly to numerical errors in the extensive and repetitive calculations required to obtain $\mathbf{BB}^T$ estimates. Particularly if the elements of the vector $\mathbf{Z}$ are highly correlated among themselves at lag zero or lag one. In such cases the only alternative is to adjust elements of $\mathbf{BB}^T$ in order to obtain the desired positive definiteness. A popular and easily used adjustment was proposed by Mejia and Millán (1974).

A new $\mathbf{BB}^T$ matrix is defined as follows:

$$\mathbf{B'B}^{T} = \mathbf{BB}^T + \lambda_j, \tag{3.23}$$

where

$$\lambda_j = \begin{bmatrix} |\lambda| & & 0 \\ & \ddots & \\ 0 & & |\lambda| \end{bmatrix},$$

and λ is the most negative eigenvalue of the original $\mathbf{BB}^T$ matrix. This adjustment is repeated, obtaining λ_j from the latest $\mathbf{B'B}'^T$, until the new $\mathbf{B'B}'^T$ matrix is positive-definite. Subscript j is used to keep track of the number of iterations.

Obtaining $\mathbf{B'}$ from $\mathbf{B'B}'^T$, the equation of the new multivariate AR(1) model will be

$$\mathbf{Z}(t) = \frac{1}{\sqrt{1 + \sum_{j=1}^{m} \lambda_j}} (\mathbf{AZ}(t-1) + \mathbf{B'}\boldsymbol{\varepsilon}(t)), \tag{3.24}$$

where m is the number of iterations needed to achieve positive semidefiniteness.

This new model preserves the mean and variance of the historical data, affecting only the correlation coefficients by a factor equal to

$$\frac{1}{\sqrt{1 + \sum_{j=1}^{m} \lambda_j}}.$$

Thus, if $\sum_{j=1}^{m} \lambda_j$ is small, the estimation error can be neglected.

3.2.2 Multivariate AR(2)

Higher-order multivariate autoregressive models are studied similarly to the AR(1). Of particular interest is the AR(2)

$$\mathbf{Z}(t) = \mathbf{AZ}(t-1) + \mathbf{BZ}(t-2) + \mathbf{C}\boldsymbol{\varepsilon}(t) \tag{3.25}$$

with parameter matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. Clarke (1973), Salas and Pegram (1977), Puente (1978), and Deeb and Puente (1979) used this model and the method-of-moments approach to identify $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. This would require defining $\mathbf{M}_0$, $\mathbf{M}_1$, and $\mathbf{M}_2$, where the latter is now the covariance matrix at lag two.

Post-multiplying Eq. (3.25) by $\mathbf{Z}^T(t)$ and taking expected values yields

$$\mathbf{M}_0 = \mathbf{AM}_1^T + \mathbf{BM}_2^T + \mathbf{CC}^T, \tag{3.26}$$

where the unit variance and white-noise properties of $\boldsymbol{\varepsilon}(t)$ have been exploited.

The lag-one covariance M_1 results from post-multiplying Eq. (3.25) by $Z^T(t-1)$ and performing expectations. It is useful to illustrate this:

$$E\left[Z(t)Z^T(t-1)\right] = AE\left[Z(t-1)Z^T(t-1)\right] + BE\left[Z(t-2)Z^T(t-1)\right]$$
$$+ CE\left[\varepsilon(t)\{Z^T(t-2)A^T + Z^T(t-3)B^T + \varepsilon^T(t-1)C^T\}\right]. \tag{3.27}$$

Using the stationary properties of covariance and again the noncorrelation of $\varepsilon(t)$ with past values of itself and of $Z(t)$ reduces the above to

$$M_1 = AM_0 + BM_1^T. \tag{3.28}$$

On post-multiplying by $Z^T(t-2)$ a similar exercise yields

$$M_2 = AM_1 + BM_0. \tag{3.29}$$

Equations (3.26), (3.28), and (3.29) are three simultaneous matrix equations on three unknowns. Combining Eqs. (3.28) and (3.29) yields

$$B = \left(M_2 - M_1 M_0^{-1} M_1\right)\left(M_0 - M_1^T M_0^{-1} M_1\right)^{-1}. \tag{3.30}$$

The resulting value of B can be used to obtain A from Eq. (3.28):

$$A = \left(M_1 - BM_1^T\right)M_0^{-1}. \tag{3.31}$$

Using A and B in Eq. (3.26) results in the quadratic form of C:

$$CC^T = M_0 - AM_1^T - BM_2^T, \tag{3.32}$$

which can be decomposed to obtain C using either of the two procedures described above.

3.2.3 Multivariate MA(1)

The form of the multivariate moving-average model of order 1 is

$$Z(t) = A\varepsilon(t) - B\varepsilon(t-1). \tag{3.33}$$

Parameters are obtained in terms of the lag-zero and lag-one covariances. Post-multiplying by $Z^T(t)$ and $Z^T(t-1)$ and taking expectations yields the following respective expressions:

$$M_0 = AA^T + BB^T \tag{3.34}$$

$$M_1 = -BA^T. \tag{3.35}$$

As in univariate moving-average models, the above two equations are nonlinear on parameters A and B. A possible solution is an iterative search as proposed by O'Connell (1974).

Equation (3.35) can be solved for B, leading to

$$B = -M_1(A^T)^{-1} \tag{3.36}$$

and

$$B^T = -A^{-1}M_1^T. \tag{3.37}$$

Substituting the above in Eq. (3.34) yields

$$AA^T + M_1(A^T)^{-1}A^{-1}M_1^T = M_0$$

or

$$AA^T + M_1(AA^T)^{-1}M_1^T = M_0. \tag{3.38}$$

With $U = AA^T$, Eq. (3.38) represents an iterative algorithm

$$U_j = M_0 - M_1 U_{j-1}^{-1} M_1^T, \tag{3.39}$$

where subscript j stands for the jth iteration. The solution starts with an assumption for U_j, possibly the identity matrix. Once U_j converges to a solution, it can be decomposed to obtain an A value, which in turn is used in Eq. (3.36) to find B. Note that A could be assumed to be lower triangular, but B will not be so.

Successful convergence of the iterative procedure ensures the preservation of M_0 and M_1 values used in the fitting exercise. O'Connell (1974) states that the convergence of such algorithms is satisfactory. In some cases though convergence may not occur and the values of U may oscillate. It is then possible to suggest a filtered solution

$$U_j = M_0 - \lambda M_1 U_{j-1}^{-1} M_1^T, \tag{3.40}$$

where λ is a value between 0 and 1. When using this equation you must keep in mind that introducing λ values different from 1 implies distortions in the preservation of M_0 and M_1 by the model.

An alternative approximate solution involves combining Eqs. (3.34) and (3.35) to yield

$$(A+B)(A+B)^T = M_0 - M_1 - M_1^T = F \tag{3.41}$$
$$(A-B)(A-B)^T = M_0 + M_1 + M_1^T = G. \tag{3.42}$$

The above are in the quadratic (BB^T) form and can be decomposed into

$$(A+B) = P \tag{3.43}$$
$$(A-B) = Q, \tag{3.44}$$

which in turn can be solved for $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{A} = \frac{\mathbf{P} + \mathbf{Q}}{2} \tag{3.45}$$

$$\mathbf{B} = \frac{\mathbf{P} - \mathbf{Q}}{2}. \tag{3.46}$$

Since $\mathbf{M}_0$ is a symmetrical, $n \times n$ matrix, and $\mathbf{M}_1$ is also $n \times n$ but not symmetrical, there are

$$n^2 + \frac{n(n+1)}{2}$$

conditions available to define matrices $\mathbf{A}$ and $\mathbf{B}$. However, the symmetry of $\mathbf{F}$ and $\mathbf{G}$ in Eqs. (3.41) and (3.42) implies that only $n(n+1)$ conditions are being imposed in the approximate solutions of $\mathbf{A}$ and $\mathbf{B}$. The implication is lack of assurance that $\mathbf{M}_1$ will in fact be exactly preserved.

Puente (1978) found that the quadratic form (Eq. 3.41) commonly is nonpositive-definite in practice, and is inconsistent with the theory and solution methods. This is because the parameters of the model are not on its feasible region. Equation (3.42) is rarely problematic, as expected from its additive nature. Modifying Eq. (3.41) using the method of Mejía and Millán implies defining

$$
\begin{aligned}
(\mathbf{A}' + \mathbf{B}')(\mathbf{A}' + \mathbf{B}')^T &= (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{B})^T + \lambda \mathbf{I} \\
&= \mathbf{F} + \lambda \mathbf{I}.
\end{aligned} \tag{3.47}
$$

Sufficient conditions for satisfying Eq. (3.42) are

$$\mathbf{A}'\mathbf{A}'^T + \mathbf{B}'\mathbf{B}'^T = \mathbf{A}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T + [\lambda/2]\mathbf{I} \tag{3.48}$$

$$\mathbf{B}'\mathbf{A}'^T = \mathbf{B}\mathbf{A}^T + [\lambda/4]\mathbf{I}, \tag{3.49}$$

which leaves Eq. (3.42) unchanged:

$$(\mathbf{A}' - \mathbf{B}')(\mathbf{A}' - \mathbf{B}')^T = (\mathbf{A} - \mathbf{B})(\mathbf{A} - \mathbf{B})^T = \mathbf{G}. \tag{3.50}$$

If and only if Eqs. (3.48) and (3.49) are satisfied, the basic multivariate MA(1) is modified to

$$\mathbf{Z}(t) = \frac{1}{\sqrt{1 + \lambda/2}} \mathbf{A}'\varepsilon(t) + \frac{1}{\sqrt{1 + \lambda/2}} \mathbf{B}'\varepsilon(t - 1), \tag{3.51}$$

which can be shown to lead to,

$$\mathbf{M}_0' = \frac{1}{1 + \lambda/2}[\mathbf{M}_0 + (\lambda/2)\mathbf{I}] \tag{3.52}$$

$$\mathbf{M}_1' = \frac{1}{1 + \lambda/2}[\mathbf{M}_1 - (\lambda/4)\mathbf{I}]. \tag{3.53}$$

If λ is small, acceptable distortions are introduced in $\mathbf{M}_0$ and $\mathbf{M}_1$, the latter suffering less.

As previously stated, the above set of equations are valid only for $\mathbf{A}'$ and $\mathbf{B}'$ obeying Eqs. (3.48) and (3.49). Note that those equations can be formulated as

$$\mathbf{A}'\mathbf{A}'^T + \mathbf{B}'\mathbf{B}'^T = \mathbf{M}_0 + (\lambda/2)\mathbf{I} \tag{3.54}$$

$$\mathbf{B}'\mathbf{A}'^T = -\mathbf{M}_1 + (\lambda/4)\mathbf{I}, \tag{3.55}$$

which in principle can be solved using an iterative scheme such as that discussed in Eqs. (3.36) to (3.39).

Usually, practitioners simply use the approximate scheme of decomposing Eqs. (3.47) and (3.50):

$$(\mathbf{A}' + \mathbf{B}') = \mathbf{P}'$$

$$(\mathbf{A}' - \mathbf{B}') = \mathbf{Q}'$$

and

$$\mathbf{A}' = \frac{\mathbf{P}' + \mathbf{Q}'}{2}$$

$$\mathbf{B}' = \frac{\mathbf{P}' - \mathbf{Q}'}{2}.$$

The above solution most probably will not satisfy Eqs. (3.48), (3.49), and (3.51). However, experience indicates that for small λs the distortions of the resulting models remain acceptable.

3.2.4 Multivariate MA(2)

The solution of

$$\mathbf{Z}(t) = \mathbf{A}\varepsilon(t) - \mathbf{B}\varepsilon(t - 1) - \mathbf{C}\varepsilon(t - 2) \tag{3.56}$$

proceeds in a manner analogous to that in the previous section. Finding $E[\mathbf{Z}(t)\mathbf{Z}^T(t)]$, $E[\mathbf{Z}(t)\mathbf{Z}^T(t-1)]$, and $E[\mathbf{Z}(t)\mathbf{Z}^T(t-2)]$ results in

$$\mathbf{M}_0 = \mathbf{A}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T + \mathbf{C}\mathbf{C}^T \tag{3.57}$$

$$\mathbf{M}_1 = -\mathbf{B}\mathbf{A}^T + \mathbf{C}\mathbf{B}^T \tag{3.58}$$

$$\mathbf{M}_2 = -\mathbf{C}\mathbf{A}^T \tag{3.59}$$

An approximate alternative to a nonlinear iterative solution of the above is to use the following valid expressions:

$$(A-B-C)(A-B-C)^T = M_0 + M_1 + M_1^T + M_2 + M_2^T \quad (3.60)$$

$$(-A-B+C)(-A-B+C)^T = M_0 + M_2 + M_2^T - M_1 - M_1^T \quad (3.61)$$

Solving for $(A-B-C) = R1$ and $(-A-B+C) = R2$ and combining yields

$$B = -\frac{R1+R2}{2}. \quad (3.62)$$

Using the resulting B in Eq. (3.57) and combining with Eq. (3.59) also implies

$$(A-C)(A-C)^T = M_0 - BB^T + M_2 + M_2^T \quad (3.63)$$

$$(A+C)(A+C)^T = M_0 - BB^T - M_2 - M_2^T, \quad (3.64)$$

which after decomposition $(A-C = R3, A+C = R4)$ can be solved for,

$$A = \frac{R3+R4}{2} \quad (3.65)$$

$$C = \frac{R4-R3}{2}. \quad (3.66)$$

Puente (1978) encountered frequent inconsistencies (negative definiteness) in Eq. (3.64). Treatment of these numerical problems using Mejia and Millán's formulation again requires the solution of equations corresponding to Eqs. (3.48) and (3.49). The resulting modified model is

$$Z(t) = \frac{1}{\sqrt{1+\lambda/2}} A'\varepsilon(t) - \frac{1}{\sqrt{1+\lambda/2}} B\varepsilon(t-1) - \frac{1}{\sqrt{1+\lambda/2}} C'\varepsilon(t-2),$$

$$(3.67)$$

where A' and C' correspond to

$$(A'+C')(A'+C')^T = (A+C)(A+C)^T + \lambda I. \quad (3.68)$$

Corresponding distortions on M_0, M_1, and M_2 are

$$M_0' = \frac{1}{(1+\lambda/2)} [M_0 + (\lambda/2)I]$$

$$M_1' = \frac{1}{(1+\lambda/2)} [-BA'^T + C'B^T]$$

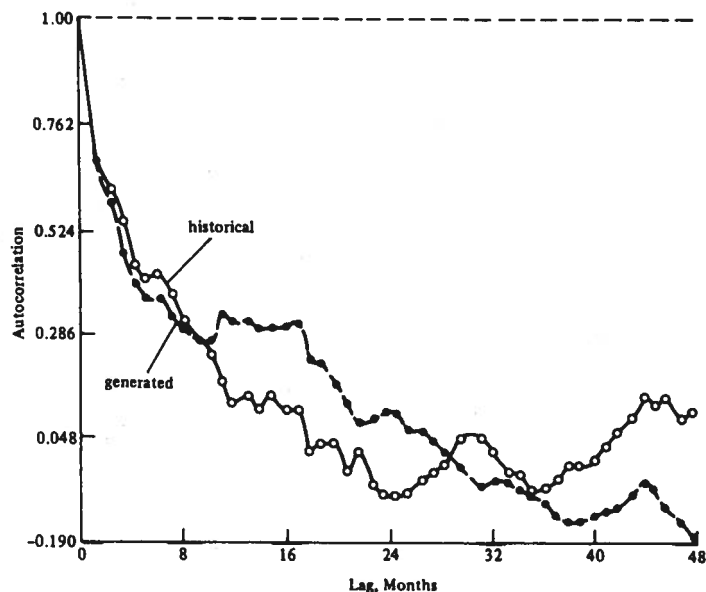$$M_2' = \frac{1}{(1+\lambda/2)} [M_2 - (\lambda/4)I].$$

Figure 3.15 Historical correlation function of station 5 versus sample correlation function from series generated with the ARMA(1,1) model, Colombian river-basin example.

Note that the approximate procedure, yielding lower triangular matrices $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$, was used for parameter estimation. The historical and generated (30 years) sample correlations are shown in Figs. 3.11 through 3.15 for all five stations. It is apparent that the overall behavior of the ARMA(1,1) is preferable to that of the AR(1) model.

3.3 SEASONAL MULTIVARIATE MODELS

3.3.1 The Seasonal Multivariate Autoregressive Model

The above example dealt with seasonality by standardizing flows but ignored nonstationarity in the correlation structure. Time-varying parameters are required to include seasonal variability in the correlation structure. In theory, it is possible to reformulate all the models in Section 3.2 in this nonstationarity mode. Most widely used is the seasonal AR(1) (Salas and Pegram, 1977; Salas et al, 1980).

The multivariate seasonal autoregressive model of order 1 will preserve all seasonal means of all variables in the state vector, all seasonal variances, all correlations among all elements of the state vector, and lag-one correlations

between adjacent seasons and between all variables. As formulated here, the model will be able to handle normal and log-normal variables as well as mixed normal and log-normal elements in the state vector.

The Multivariate Normal Case

Let

$$(\mathbf{X}_{ij} - \mathbf{m}_j) = \mathbf{A}_j(\mathbf{X}_{i,j-1} - \mathbf{m}_{j-1}) + \mathbf{B}_j \mathbf{e}_{ij}, \qquad (3.84)$$

where $\mathbf{X}_{ij}$ is the state vector ($n \times 1$) of random variables x_{ij}^{ℓ}, during the year i and season j at location ℓ with mean $\mathbf{m}_j$.

In detail,

$$\mathbf{X}_{ij} = \begin{bmatrix} x_{ij}^1 \\ x_{ij}^2 \\ \vdots \\ x_{ij}^n \end{bmatrix} \quad \text{and} \quad \mathbf{m}_j = \begin{bmatrix} m_{x_j}^1 \\ m_{x_j}^2 \\ \vdots \\ m_{x_j}^n \end{bmatrix}.$$

For example, x_{ij}^1 can be interpreted as the monthly streamflow at station 1 during year i and month j (season) and $m_{x_j}^1$ as the mean value of this variable during the month j.

$\mathbf{A}_j$ and $\mathbf{B}_j$ are parameter matrices ($n \times n$), one for each season. The ($n \times 1$) vector of standard normal deviates is $\mathbf{e}_{ij}$, for year i and season j.

Notation for Eq. (3.84) is simplified by introducing the zero-mean vector, $\mathbf{Z}_{ij} = (\mathbf{X}_{ij} - \mathbf{m}_j)$:

$$\mathbf{Z}_{ij} = \mathbf{A}_j \mathbf{Z}_{i,j-1} + \mathbf{B}_j \mathbf{e}_j. \qquad (3.85)$$

The similarity between Eqs. (3.85) and (3.1) is obvious; the only difference is the seasonal dependence of $\mathbf{A}_j$ and $\mathbf{B}_j$. The lag-zero covariance for season j is $_j\mathbf{M}_0$. The covariance matrix between vectors $\mathbf{Z}_{ij}$ and $\mathbf{Z}_{i,j-1}$ is $_j\mathbf{M}_1$. Following a derivation analogous to that of Section 3.2.1, it should be easy to arrive at the following results:

$$\mathbf{A}_j = {}_j\mathbf{M}_{1\ j-1}\mathbf{M}_0^{-1} \qquad (3.86)$$

$$\mathbf{B}_j\mathbf{B}_j^T = {}_j\mathbf{M}_0 - {}_j\mathbf{M}_{1\ j-1}\mathbf{M}_0^{-1}{}_j\mathbf{M}_1^T. \qquad (3.87)$$

The decomposition of $\mathbf{B}_j\mathbf{B}_j^T$ is accomplished using the procedures described in Section 3.2.1. Since every covariance function is seasonally dependent, so are the resulting parameters.

It is now convenient notationally to redefine the vectors Z_{ij} and $Z_{i,j-1}$ as Y and X, respectively. That is, vector Y, with n elements y_i, $i = 1, \ldots, n$, will represent the state vector at season j; and vector X, with elements x_i, $i = 1, \ldots, n$, will represent the state vector in the previous season $j - 1$. The model is now

$$Y = A_j X + B_j \varepsilon_j. \tag{3.88}$$

The covariances of interest are now,

$$_j M_0 = S_{yy} = E[YY^T]$$

$$_j M_1 = S_{yx} = E[YX^T]$$

$$_{j-1} M_0 = S_{xx} = E[XX^T]$$

$$S_{xy} = S_{yx}^T$$

Matrices S_{xx}, S_{xy}, S_{yx}, and S_{yy} can be represented in terms of variances, standard deviations, and correlations as

$$S_{xx} = \begin{bmatrix} S_{x_1}^2 & r_{x_1 x_2} S_{x_1} S_{x_2} & \cdots & r_{x_1 x_n} S_{x_1} S_{x_n} \\ r_{x_2 x_1} S_{x_2} S_{x_1} & S_{x_2}^2 & & r_{x_2 x_n} S_{x_2} S_{x_n} \\ \vdots & & \ddots & \vdots \\ r_{x_n x_1} S_{x_n} S_{x_1} & r_{x_n x_2} S_{x_n} S_{x_2} & \cdots & S_{x_n}^2 \end{bmatrix} \tag{3.89}$$

$$S_{yy} = \begin{bmatrix} S_{y_1}^2 & r_{y_1 y_2} S_{y_1} S_{y_2} & & r_{y_1 y_n} S_{y_1} S_{y_n} \\ \vdots & \ddots & & \vdots \\ r_{y_n y_1} S_{y_n} S_{y_1} & r_{y_n y_2} S_{y_n} S_{y_2} & \cdots & S_{y_n}^2 \end{bmatrix} \tag{3.90}$$

$$S_{yx} = \begin{bmatrix} r_{y_1 x_1} S_{x_1} S_{y_1} & r_{y_2 x_1} S_{x_1} S_{y_2} & \cdots & r_{y_n x_1} S_{x_1} S_{y_n} \\ \vdots & \ddots & & \vdots \\ r_{y_1 x_n} S_{x_n} S_{y_1} & r_{y_2 x_n} S_{x_n} S_{y_2} & \cdots & r_{y_n x_n} S_{x_n} S_{y_n} \end{bmatrix}, \tag{3.91}$$

where S_{x_i} is the standard deviation of variable x_i, $r_{x_i x_j}$ is the lag-zero correlation between stations (variables) x_i and x_j, and $r_{y_i x_j}$ is the lag-one correlation between variables y_i and x_j.

The estimation of sample covariances should again follow Eqs. (3.12) and (3.13) to minimize the occurrences of inconsistent (nonpositive definite) $B_j B_j^T$ matrices.

Multivariate Log-Normal Case

Consider the case where the elements of vectors X and Y, x_i and y_i, are random variables following a two-parameter log-normal distribution. Define the new variables, x_i' and y_i', as follows:

$$x_i' = \ln(x_i) \tag{3.92}$$

$$y_i' = \ln(y_i). \tag{3.93}$$

Thus, the original variables x_i and y_i are log-normally distributed with means m_{x_i} and m_{y_i}, standard deviations S_{x_i} and S_{y_i}, and the correlation coefficient among them given by $r_{x_i y_i}$.

Therefore, the transformed variables x_i' and y_i' are normally distributed with means $m_{x_i'}$ and $m_{y_i'}$, standard deviations $S_{x_i'}$ and $S_{y_i'}$, and the correlation coefficient among them equal to $r_{x_i' y_i'}$.

To preserve without bias the statistics of the original variables instead of the statistics of the transformed variables it is necessary to compute parameters of the distribution of x_i' and y_i' based on the parameters of the distribution of the original variables x_i and y_i, using the expressions given by Matalas (1967):

$$m_{x_i} = \exp\left\{ S_{x_i'}^2/2 + m_{x_i'} \right\}$$

$$m_{y_i} = \exp\left\{ S_{y_i'}^2/2 + m_{y_i'} \right\}$$

$$S_{x_i}^2 = \exp\left\{ 2\left[S_{x_i'}^2 + m_{x_i'} \right] \right\} - \exp\left[S_{x_i'}^2 + 2 m_{x_i'} \right]$$

$$S_{y_i}^2 = \exp\left\{ 2\left[S_{y_i'}^2 + m_{y_i'} \right] \right\} - \exp\left[S_{y_i'}^2 + 2 m_{y_i'} \right]$$

$$r_{x_i y_i} = \frac{\exp\left\{ S_{x_i'} S_{y_i'} r_{x_i' y_i'} \right\} - 1}{\sqrt{\left\{ \exp(S_{x_i'}^2) - 1 \right\} \left\{ \exp(S_{y_i'}^2) - 1 \right\}}}.$$

Solving the above system of simultaneous equations for $m_{x_i'}$, $m_{y_i'}$, $S_{x_i'}$, $S_{y_i'}$, and $r_{x_i' y_i'}$ yields

$$m_{x_i'} = \ln(m_{x_i}) - S_{x_i'}^2/2 \tag{3.94}$$

$$S_{x_i'}^2 = \ln\left\{ \frac{S_{x_i}^2}{m_{x_i}^2} + 1 \right\} \tag{3.95}$$

$$m_{y_i'} = \ln(m_{y_i}) - S_{y_i'}^2/2 \tag{3.96}$$

$$S_{y_i'}^2 = \ln\left\{ \frac{S_{y_i}^2}{m_{y_i}^2} + 1 \right\} \tag{3.97}$$

$$r_{x_i' y_i'} = \frac{\ln\left\{ 1 + r_{x_i y_i} \sqrt{(e^{S_{x_i}} - 1)(e^{S_{y_i}} - 1)} \right\}}{S_{x_i'} S_{y_i'}}. \tag{3.98}$$

Thus, having the sample variance and correlation coefficient from the historical records, it is possible to compute the values of the transformed variables using Eqs. (3.94) through (3.98).

The parameters of the transformed variables can be used to build the necessary autocovariance and cross-covariance matrices using the definitions given in Eqs. (3.89), (3.90), and (3.91). Generation matrices A_j and B_j are then available from the previously derived equations.

In order to get the original variables from generated synthetic data, the user must perform the inverse transformation

$$x_i = \exp\left(x_i'\right)$$

$$y_i = \exp\left(y_i'\right).$$

Mixture of Normal and Log-Normal Variables in the Autoregressive Model

Variables such as streamflow are best described by different marginal distributions at each season. For example, the Nile River (Curry and Bras, 1978) exhibits alternating normal and log-normal distributions each month of the year. The multivariate autoregressive model allows the mixing of normal and log-normal variables.

Let x_i and y_i be the original random variables. Assume x_i is log-normally distributed and y_i is normally distributed:

$$x_i' = \ln\left(x_i\right)$$

$$x_i' \sim N\left(m_{x_i}, S_{x_i}^2\right)$$

$$y_i \sim N\left(m_{y_i}, S_{y_i}^2\right).$$

The procedure to obtain the parameters m_{x_i} and $S_{x_i}^2$ of the log-normal variables was described in the last section and made use of Eqs. (3.94) to (3.98). In this case, the problem is to obtain the correlation coefficient between the log-normal and the normal variables while preserving the parameters of the untransformed data. The necessary expression is given by Mejia et al. (1974).

If $r_{x_i y_j}$ is the correlation coefficient among the original variables and $r_{x_i' y_j}$ the correlation coefficient between the transformed log-normal and the normal variables, the following relationship holds,

$$r_{x_i' y_j} = \frac{r_{x_i y_j}\sqrt{\left(e^{S_{x_i}^2} - 1\right)}}{S_{x_i'}}.$$

(3.99)

In order to illustrate how to build the required covariance matrices, consider the following example.

Define the random vectors X and Y with three random variables each,

$$X = \begin{bmatrix} x_1 \\ x_2' \\ x_3' \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3' \end{bmatrix}, \qquad (3.100)$$

where the random variables x_1, y_1, y_2 are normally distributed, and x_2', x_3', y_3' are log-normally distributed. Then the corresponding covariance matrices will be

$$S_{xx} = \begin{bmatrix} \left(S_{x_1}\right)^2 & r_{x_1 x_2'}S_{x_1}S_{x_2'} & r_{x_1 x_3'}S_{x_1}S_{x_3'} \\ r_{x_2' x_1}S_{x_2'}S_{x_1} & \left(S_{x_2'}\right)^2 & r_{x_2' x_3'}S_{x_2'}S_{x_3'} \\ r_{x_3' x_1}S_{x_3'}S_{x_1} & r_{x_3' x_2'}S_{x_3'}S_{x_2'} & \left(S_{x_3'}\right)^2 \end{bmatrix}$$

$$S_{yy} = \begin{bmatrix} \left(S_{y_1}\right)^2 & r_{y_1 y_2}S_{y_1}S_{y_2} & r_{y_1 y_3'}S_{y_1}S_{y_3'} \\ r_{y_2 y_1}S_{y_2}S_{y_1} & \left(S_{y_2}\right)^2 & r_{y_2 y_3'}S_{y_2}S_{y_3'} \\ r_{y_3' y_1}S_{y_3'}S_{y_1} & r_{y_3' y_2}S_{y_3'}S_{y_2} & \left(S_{y_3'}\right)^2 \end{bmatrix} \qquad (3.101)$$

$$S_{xy} = \begin{bmatrix} r_{x_1 y_1}S_{x_1}S_{y_1} & r_{x_1 y_2}S_{x_1}S_{y_2} & r_{x_1 y_3'}S_{x_1}S_{y_3'} \\ r_{x_2' y_1}S_{x_2'}S_{y_1} & r_{x_2' y_2}S_{x_2'}S_{y_2} & r_{x_2' y_3'}S_{x_2'}S_{y_3'} \\ r_{x_3' y_1}S_{x_3'}S_{y_1} & r_{x_3' y_2}S_{x_3'}S_{y_2} & r_{x_3' y_3'}S_{x_3'}S_{y_3'} \end{bmatrix}$$

The above matrices are again routinely used in the existing expressions to obtain A_j and B_j. The user must remember to add means and perform required inverse transformations at the end of the computations. Since the elements of the sample covariance matrices are now individually evaluated and result from different nonlinear transformations, it is not unusual to obtain inconsistent estimates for the $B_j B_j^T$ matrix.

3.4 A GENERAL MULTIVARIATE SEASONAL STREAMFLOW MODEL

This section generalizes seasonal multivariate streamflow models. In contrast to the well-defined structure discussed in the past sections, the model presented here can be manipulated to a variety of forms. In the simplest case it will collapse to the multivariate AR(1) model previously seen.

This presentation follows the work of Curry and Bras (1980) and in a sense is a generalization to the multivariate case of the univariate model suggested by Rao and Kashyap (1973, 1974) and Kashyap and Rao (1976).

3.4.1 General Model Form and Identification

The structure of the general model is

$$
\mathbf{Y}(k,i) = \mathbf{\Phi}_i \begin{bmatrix} \mathbf{Y}(k,i-1) \\ \mathbf{Y}(k,i-2) \\ \vdots \\ \mathbf{Y}(k,i-n_1(i)) \end{bmatrix} + \mathbf{U}(i) + \mathbf{V}(k,i) + \mathbf{\Gamma}_i \begin{bmatrix} \mathbf{V}(k,i-1) \\ \mathbf{V}(k,i-2) \\ \vdots \\ \mathbf{V}(k,i-n_2(i)) \end{bmatrix},
$$

$$(3.102)$$

where $\mathbf{Y}(k,i)$ is a vector of streamflows (or other stochastic variables) at year k, season i at n_0 locations or gaging stations

$$
\mathbf{Y}(k,i) = \begin{bmatrix} Y_1(k,i) \\ Y_2(k,i) \\ \vdots \\ Y_{n_0}(k,i) \end{bmatrix}.
$$

The $(n_0 \times 1)$ vector $\mathbf{U}(i)$ represents a deterministic term to account for seasonal variation in the mean of the process. This term is periodic, with a basic cycle of one year in the case of streamflows.

The $(n_0 \times 1)$ vector $\mathbf{V}(k,i)$ represents a random component or noise

$$
\mathbf{V}(k,i) = \begin{bmatrix} V_1(k,i) \\ V_2(k,i) \\ \vdots \\ V_{n_0}(k,i) \end{bmatrix}.
$$

Each of the elements of the above vector can be correlated. Correlation is introduced by operating on uncorrelated white noise to form each $V_j(k,i)$

$$
V_j(k,i) = \sum_{m=1}^{n_0} C_i(j,m) W_m(k,i),
$$

where $C_i(j,m)$ is an element of an $(n_0 \times n_0)$ matrix of coefficients, and $W_m(k,i)$ is a white-noise input at station m, year k, season i.

Matrix $\mathbf{\Gamma}_i$, of dimensions $n_0 \times n_0 n_2(i)$, induces autocorrelation of noise terms affecting the present discharge at all locations.

Matrix $\mathbf{\Phi}_i$, with dimensions $n_0 \times n_0 n_1(i)$, controls the transitions to the month i discharges from past occurrences.

The above model is clearly not parsimonious. Besides the obvious parameter matrices $\mathbf{\Phi}_i$, $\mathbf{\Gamma}_i$, and vector $\mathbf{U}(i)$ (note the seasonal dependence of all of the above), you must also fix the autoregressive lag $n_1(i)$, the noise lag $n_2(i)$, and

matrix C_i, for all seasons. However, as will be seen, the estimation of all these parameters is not necessary or is simplified in many cases.

The peculiarity of the model represented in Eq. (3.102) is that each station or location is potentially dependent on every other station at several time lags. The model allows for seasonally varying mean, variances, and correlation structures. Finally, noise terms are not only cross-correlated between stations but may be autocorrelated through the introduction of a moving average of past shocks.

A preliminary identification of structure and parameters is sometimes possible. Such decisions should be guided by knowledge of the process being modeled. The concept of causality may be very valuable in this step. Granger and Newbold (1977) discuss it in detail; Curry and Bras (1980) illustrate it together with the model being studied here.

Essentially, in most geophysical and natural processes, it is valid to state that the future cannot cause the past.

Using the above concepts, it may be possible to determine *a priori* that many of the elements in matrix $\mathbf{\Phi}_i$ are zero. For example, in a river basin, flow in upstream stations cannot be caused by past flows in downstream stations. Nevertheless, if the set of possibly causal variables is not complete, spurious relationships may occur. For example, past downstream stations may show some explanatory power relative to upstream streamflow stations if they are in any way acting as surrogates for past rainfall on upstream locations. Care must then be exercised in interpreting causality.

When using the above model for streamflow, it is also possible to state that $\mathbf{U}(i)$ has a yearly period and to limit $n_1(i)$ and $n_2(i)$ to a reasonable range. Structural simplifications are also possible by eliminating the autocorrelated residuals, i.e., using a model with $\mathbf{\Gamma}_i$ equal to zero. In a forecasting application it will be seen that the need to estimate C_i does not arise.

Another structural simplification arises by assuming that matrices $\mathbf{\Phi}_i$, $\mathbf{\Gamma}_i$, and parameters $n_1(i)$ and $n_2(i)$ are in fact seasonally invariant, and not a function of i.

The estimation of the nonautocorrelated and constant parameter models are the subject of the next subsections.

3.4.2 Parameter Estimation

Two possible simplifications of the generalized model were suggested previously. One implied elimination of the autocorrelation of residuals, leading to the following form,

$$
\mathbf{Y}(k,i) = \mathbf{\Phi}_i \begin{bmatrix} \mathbf{Y}(k,i-1) \\ \mathbf{Y}(k,i-2) \\ \vdots \\ \mathbf{Y}(k,i-n_1(i)) \end{bmatrix} + \mathbf{U}(i) + \mathbf{V}(k,i) \qquad \begin{array}{l} k=0,1,\ldots, \\ i=0,1,\ldots,11, \end{array}
$$

$$(3.103)$$

where for simplicity of discussion a season has been considered equal to a month.

The other alternative was to assume seasonally invariant parameter matrices,

$$\mathbf{Y}(k,i) = \Phi \begin{bmatrix} \mathbf{Y}(k,i-1) \\ \mathbf{Y}(k,i-2) \\ \vdots \\ \mathbf{Y}(k,i-n_1) \end{bmatrix} + \mathbf{U}(i) + \mathbf{V}(k,i) + \Gamma \begin{bmatrix} \mathbf{V}(k,i-1) \\ \mathbf{V}(k,i-2) \\ \vdots \\ \mathbf{V}(k,i-n_2) \end{bmatrix} \quad (3.104)$$

$$k = 0,1,\dots,$$
$$i = 0,1,2,\dots,11$$

Parameter Estimation of the Nonautocorrelated Residual Model

Parameter estimation in this case relies heavily on Generalized Least Squares techniques. Excellent and complete references on the subject are Goldberger (1964) and Johnston (1972), and the reader is referred there for review, if necessary. The following discussion assumes knowledge of linear regression and closely follows Curry and Bras (1980).

From Eq. (3.103), the expression for any particular station j and month i is

$$Y_j(k,i) = \sum_{\ell=1}^{n_1(i)} \sum_{m=1}^{n_0} \left[\Phi_i(j,(m-1)n_1(i)+\ell) \cdot Y_m(k,i-\ell) \right]$$
$$+ U_j(i) + V_j(k,i). \quad (3.105)$$

As previously discussed, some elements of Φ_i may be assumed *a priori* to be zero. However, to simplify the following discussion, it will be assumed that this information is not available.

If there are N total months of observations, or $N/12$ observations of month i, Eq. (3.105) may be written in the notation of a general linear model [let $n_1(i) = 1$ and $j = 1$ for ease of notation]:

$$\mathbf{Z}_1 = \mathbf{X}_1 \boldsymbol{\beta}_1 + \boldsymbol{\varepsilon}_1, \quad (3.106)$$

where

$$\mathbf{Z}_1 = \begin{bmatrix} Y_1(1,i) \\ Y_1(2,i) \\ \vdots \\ Y_1(N/12,i) \end{bmatrix}$$

$$\mathbf{X}_1 = \begin{bmatrix} 1 & Y_1(1,i-1) & Y_2(1,i-1) & \cdots & Y_{n_0}(1,i-1) \\ 1 & Y_1(2,i-1) & Y_2(2,i-1) & \cdots & Y_{n_0}(2,i-1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & Y_1(N/12,i-1) & Y_2(N/12,i-1) & \cdots & Y_{n_0}(N/12,i-1) \end{bmatrix}$$

$$\boldsymbol{\beta}_1 \cong \begin{bmatrix} U_1(i) \\ \Phi_i(1,1) \\ \Phi_i(1,2) \\ \vdots \\ \Phi_i(1,n_0) \end{bmatrix}$$

$$\boldsymbol{\varepsilon}_1 = \begin{bmatrix} V_1(1,i) \\ V_1(2,i) \\ \vdots \\ V_1(N/12,i) \end{bmatrix}.$$

It would be possible to use Ordinary Least Squares techniques to estimate the coefficients in Eq. (3.105) (as represented in Eq. 3.106) if the error term between different equations (locations) were not correlated, i.e., $E[V_j(k,i)V_\ell(k,i)] = 0, \forall j, \ell$. In fact, by construction, they will undoubtedly be correlated. In such cases, Ordinary Least Squares estimators, although consistent, are not asymptotically efficient. The coefficients of a single equation of the multivariate model (Eq. 3.105) should be estimated simultaneously with all other coefficients. This may be accomplished as follows.

The $N/12$ observations of each equation of the multivariate model may be written together in the notation of a general linear model as

$$\begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \\ \vdots \\ \mathbf{Z}_{n_0} \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1 & & & \mathbf{0} \\ & \mathbf{X}_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{X}_{n_0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \\ \vdots \\ \boldsymbol{\beta}_{n_0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \boldsymbol{\varepsilon}_2 \\ \vdots \\ \boldsymbol{\varepsilon}_{n_0} \end{bmatrix}, \quad (3.107)$$

where $\mathbf{Z}_i$, $\mathbf{X}_i$, and $\boldsymbol{\varepsilon}_i$, $i = 1,2,\dots,n_0$, are as defined in Eq. (3.106). Equation (3.107) may be written as

$$\mathbf{Z}' = \mathbf{X}' \boldsymbol{\beta}' + \boldsymbol{\varepsilon}'. \quad (3.108)$$

The error term in a single station equation (Eq. 3.106) is homoscedastic (constant variance) and not cross-correlated, implying

$$E\left[\boldsymbol{\varepsilon}_i \boldsymbol{\varepsilon}_i^T \right] = \sigma_{\varepsilon_i \varepsilon_i} \mathbf{I}$$

$$E\left[\boldsymbol{\varepsilon}_i \boldsymbol{\varepsilon}_j^T \right] = \sigma_{\varepsilon_i \varepsilon_j} \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix. By definition, the variance–covariance matrix of ε' is then,

$$E[\varepsilon'\varepsilon'^T] = \begin{bmatrix} E[\varepsilon_1\varepsilon_1^T] & E[\varepsilon_1\varepsilon_2^T] & \cdots & E[\varepsilon_1\varepsilon_{n_0}^T] \\ E[\varepsilon_2\varepsilon_1^T] & E[\varepsilon_2\varepsilon_2^T] & \cdots & E[\varepsilon_2\varepsilon_{n_0}^T] \\ \vdots & \vdots & & \vdots \\ E[\varepsilon_{n_0}\varepsilon_1^T] & E[\varepsilon_{n_0}\varepsilon_2^T] & \cdots & E[\varepsilon_{n_0}\varepsilon_{n_0}^T] \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{\varepsilon_1\varepsilon_1} & \sigma_{\varepsilon_1\varepsilon_2} & \cdots & \sigma_{\varepsilon_1\varepsilon_{n_0}} \\ \sigma_{\varepsilon_2\varepsilon_1} & \sigma_{\varepsilon_2\varepsilon_2} & \cdots & \sigma_{\varepsilon_2\varepsilon_{n_0}} \\ \vdots & \vdots & & \vdots \\ \sigma_{\varepsilon_{n_0}\varepsilon_1} & \sigma_{\varepsilon_{n_0}\varepsilon_2} & \cdots & \sigma_{\varepsilon_{n_0}\varepsilon_{n_0}} \end{bmatrix} \otimes \mathbf{I} \equiv \mathbf{\Sigma}_{\varepsilon'\varepsilon'} \otimes \mathbf{I}, \quad (3.109)$$

where $\otimes$ denotes Kronecker multiplication of matrices.*

The homoscedasticity and noncorrelation necessary for Ordinary Least Squares (OLS) are then violated by the complete model. However, using Generalized Least Squares (GLS) estimators, the coefficients of Eq. (3.107) can be consistently and efficiently estimated.

Let

$$E[\varepsilon'\varepsilon'^T] = \mathbf{\Omega},$$

where $\mathbf{\Omega}$ is the matrix defined in Eq. (3.109). Assuming that $\mathbf{\Omega}$ is a positive-definite matrix, there exists a nonsingular $\mathbf{P}$ that satisfies

$$\mathbf{\Omega} = \mathbf{P}\mathbf{P}^T$$

so that

$$\mathbf{P}^{-1}\mathbf{\Omega}(\mathbf{P}^{-1})^T = \mathbf{I}$$

and

$$(\mathbf{P}^{-1})^T\mathbf{P}^{-1} = \mathbf{\Omega}^{-1}.$$

---

*

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1m}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2m}\mathbf{B} \\ \vdots & \vdots & & \vdots \\ a_{n1}\mathbf{B} & a_{n2}\mathbf{B} & \cdots & a_{nm}\mathbf{B} \end{bmatrix}$$

where $\mathbf{A}$ is an $(n \times m)$ matrix and $\mathbf{B}$ is of arbitrary dimension.

Pre-multiplying Eq. (3.108) by $\mathbf{P}^{-1}$ gives

$$\mathbf{P}^{-1}\mathbf{Z}' = \mathbf{P}^{-1}\mathbf{X}'\mathbf{\beta}' + \mathbf{P}^{-1}\varepsilon'$$

or

$$\mathbf{Z}'' = \mathbf{X}''\mathbf{\beta}' + \varepsilon'', \quad (3.110)$$

where

$$\mathbf{Z}'' = \mathbf{P}^{-1}\mathbf{Z}'$$
$$\mathbf{X}'' = \mathbf{P}^{-1}\mathbf{X}'$$
$$\varepsilon'' = \mathbf{P}^{-1}\varepsilon'.$$

The covariance of ε'' is given by

$$E[\varepsilon''\varepsilon''^T] = \mathbf{P}^{-1}\mathbf{\Omega}(\mathbf{P}^{-1})^T$$
$$= \mathbf{I}.$$

Thus, the transformed model of Eq. (3.110) satisfies the assumptions required for OLS estimation so that

$$\mathbf{\beta}' = (\mathbf{X}''^T\mathbf{X}'')^{-1}\mathbf{X}''^T\mathbf{Z}''$$
$$= (\mathbf{X}'^T\mathbf{\Omega}^{-1}\mathbf{X}')^{-1}\mathbf{X}'^T\mathbf{\Omega}^{-1}\mathbf{Z}' \quad (3.111)$$

is a consistent and efficient estimator of the coefficients $\mathbf{\beta}'$. A consistent estimate of the variance–covariance matrix of the estimated coefficients is given by Goldberger (1964)

$$\mathbf{S}_{\mathbf{\beta}'\mathbf{\beta}'} = E\left[(\mathbf{\beta}' - \mathbf{\beta}')(\mathbf{\beta}' - \mathbf{\beta}')^T\right] = [\mathbf{X}'^T\mathbf{\Omega}^{-1}\mathbf{X}']^{-1}. \quad (3.112)$$

The difficulty with implementing Eqs. (3.111) and (3.112) is that the matrix $\mathbf{\Omega}^{-1}$ is unknown. However, the OLS estimator may be used on each individual station equation (Eq. 3.106) of the multivariate model to obtain estimates of the coefficients (Johnston, 1972). Using these coefficients, the residuals of each individual equation may be computed and subsequently used to estimate $\mathbf{\Omega}^{-1}$. From Eq. (3.109)

$$\hat{\mathbf{\Omega}} = \mathbf{S}_{\varepsilon'\varepsilon'} \otimes I. \quad (3.113)$$

where $\mathbf{S}_{\varepsilon'\varepsilon'}$ is an estimate of $\mathbf{\Sigma}_{\varepsilon'\varepsilon'}$. A given element of $\mathbf{S}_{\varepsilon'\varepsilon'}$ is estimated from

$$S_{\varepsilon'\varepsilon'}(i, j) = \frac{(\mathbf{Z}_i - \mathbf{X}_i\mathbf{\beta}_i)^T(\mathbf{Z}_j - \mathbf{X}_j\mathbf{\beta}_j)}{((N/12) - \text{rank}(\mathbf{X}_i))^{1/2}((N/12) - \text{rank}(\mathbf{X}_j))^{1/2}}, \quad (3.114)$$

where $\hat{\beta}$ is the OLS estimate of β, and Z and X are as defined in Eq. (3.106). Thus, using Eqs. (3.113) and (3.114), $\hat{\Omega}$ may be found.

Parameter Estimation of the Autocorrelated Residuals Model

The expression of one element (one station) of the vector $Y(k, i)$ in Eq. (3.104) can be expressed in terms of a continuous index t, by assuming a 12-season (months) year:

$$Y_j(t) = \sum_{\ell=1}^{n_1} \sum_{m=1}^{n_0} \Phi(j, (m-1)n_0 + \ell) Y_m(t-\ell) + U_j(\text{mod}_{12}(t))$$

$$+ V_j(t) + \sum_{\ell=1}^{n_2} \sum_{m=1}^{n_0} \Gamma(j, (m-1)n_0 + \ell) V_m(t-\ell), \qquad (3.115)$$

where $t = 12k + i$; $k = 0, 1, 2, \ldots$; $i = 0, 1, 2, \ldots, 11$; and $\text{mod}_{12}(t)$ is the modulus, base 12, of t.

Keep in mind that the terms U_j and V_m remain seasonally dependent. In fact $U_j(i)$ can be expressed as

$$U_j(i) = \alpha(j, 1) + \sum_{\ell=1}^{n_3} \left[\alpha(j, 2\ell) \cos \omega_\ell(i+1) + \alpha(j, 2\ell+1) \sin \omega_\ell(i+1) \right]$$

$$(3.116)$$

$$i = 0, 1, \ldots, 11; \quad \omega_\ell = \frac{2\pi\ell}{12}; \quad n_3 \leq 6$$

The use of OLS estimators with Eq. (3.115) is inappropriate because the explanatory variables $Y_m(t-\ell)$ are contemporaneously correlated with the disturbance terms.

Panuska (1969) derived a consistent estimation algorithm for the coefficients of a particular case of the stochastic difference equation given in Eq. (3.115). The algorithm was developed for the univariate case (i.e., only one Y sequence) and for $V(\cdot)$ with constant variance. In Kashyap (1971), Kashyap and Rao (1976), and Rao and Kashyap (1973, 1974), theoretical properties of the algorithm are discussed in great detail.

An algorithm, essentially like Panuska's, but capable of estimating the parameters of the multivariate Eq. (3.104) is derived in the following paragraphs. The algorithm is closely related to GLS estimation as discussed in the past section.

The single element Eq. (3.115) of the multivariate model can be written in the form of a linear model (for convenience, let $n_1 = 1$ and $n_2 = 1$, and assume that no elements of Φ or Γ are *a priori* assumed zero)

$$Z_j(t) = X_j(t)\beta_j + \varepsilon_j(t), \qquad (3.117)$$

where

$$Z_j = Y_j(t)$$

$$\varepsilon_j(t) = V_j(t)$$

$$X_j(t) = \Big[Y_1(t-1) Y_2(t-1), \ldots, Y_{n_0}(t-1), 1,$$

$$\cos \omega_1(\text{mod}_{12}(t)), \sin \omega_1(\text{mod}_{12}(t)), \ldots,$$

$$\cos \omega_{n_3}(\text{mod}_{12}(t)), \sin \omega_{n_3}(\text{mod}_{12}(t)),$$

$$\varepsilon_1(t-1), \varepsilon_2(t-1), \ldots, \varepsilon_{n_0}(t-1) \Big]$$

$$\beta_j^T = \Big[\Phi(j, 1), \Phi(j, 2), \ldots, \Phi(j, n_0), \alpha(j, 1), \alpha(j, 2), \ldots,$$

$$\alpha(j, 2n_3 + 1), \Gamma(j, 1), \Gamma(j, 2), \ldots, \Gamma(j, n_0) \Big].$$

Given N observations of $Y_j(\cdot)$ $\{Y_j(i), i = 0, 1, 2, \ldots, N-1\}$, the station equation may be written as

$$Z_j^N = X_j^N \beta_j + \varepsilon_j^N, \qquad (3.118)$$

where

$$Z_j^N = \begin{bmatrix} Z_j(0) \\ Z_j(1) \\ \vdots \\ Z_j(N-1) \end{bmatrix} = \begin{bmatrix} Y_j(0) \\ Y_j(1) \\ \vdots \\ Y_j(N-1) \end{bmatrix}$$

$$X_j^N = \begin{bmatrix} X_j(0) \\ X_j(1) \\ \vdots \\ X_j(N-1) \end{bmatrix}$$

$$\varepsilon_j^N = \begin{bmatrix} \varepsilon_j(0) \\ \varepsilon_j(1) \\ \vdots \\ \varepsilon_j(N-1) \end{bmatrix}$$

The superscript N is used to call attention to the number of observations, or row dimension, of each vector or matrix. This notation will simplify the subsequent derivation.

The variance of $\varepsilon_j(t)$, $V_j(t)$, is a function of season and hence the variance of $\varepsilon_j(\cdot)$ has period 12 when modeling monthly discharges. For now, assume that

$$\mathrm{var}\left[\varepsilon_j(t)\right] = \Psi_j(\mathrm{mod}_{12}(t))^2 \qquad (3.119)$$

Using Eq. (3.119) and noting that $\varepsilon_j(\cdot)$ is not autocorrelated (i.e., $E[\varepsilon_j(i)\varepsilon_j(\ell)] = 0 \ \forall i \neq \ell$) the noise covariance matrix is

$$E\left[\varepsilon_j^N(\varepsilon_j^N)^T\right] = \begin{bmatrix} \Psi_j(0)^2 & & & & & & 0 \\ & \Psi_j(1)^2 & & & & & \\ & & \ddots & & & & \\ & & & \Psi_j(11)^2 & & & \\ & & & & \Psi_j(0)^2 & & \\ & & & & & \ddots & \\ 0 & & & & & & \Psi_j(\mathrm{mod}_{12}(N))^2 \end{bmatrix}$$

$$= \Omega_j^N. \qquad (3.120)$$

The use of the Generalized Least Squares estimator of β_j in Eq. (3.118) will yield consistent estimates since all elements of X_j^N are contemporaneously uncorrelated with ε_j^N (each sequence $\{V_j(i), i = 0, 1, \ldots, N-1\} \ \forall j$ is a white-noise process) and other necessary assumptions are satisfied. Thus,

$$\beta_j^N = \left((X_j^N)^T(\Omega_j^N)^{-1}X_j^N\right)^{-1}(X_j^N)^T(\Omega_j^N)^{-1}Z_j^N. \qquad (3.121)$$

Of course this does not solve the estimation problem since the $\varepsilon(\cdot)$s (and hence X_j^N) and $\Psi_j(\cdot)$s (and hence Ω_j^N) are unknown.

Note, however, that given initial estimates of parameter vectors $\{\beta_j^1, j = 1, 2, \ldots, n_0\}$ the residuals $\{\hat{\varepsilon}_j(1), j = 1, 2, \ldots, n_0\}$ may be estimated as

$$\hat{\varepsilon}_j(1) = Z_j(1) - X_j(1)\beta_j^1 \qquad j = 1, 2, \ldots, n_0, \qquad (3.122)$$

where the elements $\{\varepsilon_j(0), j = 1, 2, \ldots, n_0\}$ in $X_j(1)$ are replaced with their expected value (zero). The estimated residuals $\{\hat{\varepsilon}_j(1), j = 1, 2, \ldots, n_0\}$ may then be used to form estimated matrices $\{X_j, j = 1, 2, \ldots, n_0\}$. Assuming that estimates exist of the functions $\{\Psi_j(\cdot), j = 1, 2, \ldots, n_0\}$ (this point will be discussed later) the matrices $\{\Omega_j, j = 1, 2, \ldots, n_0\}$ may be formed and β_j for all j may be estimated using Eq. (3.121). This process can be repeated until the parameters $\{\hat{\beta}_j, j = 1, 2, \ldots, n_0\}$ are found after N iterations. Keep in mind

that although not explicitly indicated, matrices X_j, Ω_j^i, and S_j^i are estimates in Eqs. (3.121), (3.122), and, following expressions.

As formulated, the above estimation algorithm is computationally inefficient. The addition of a new observation requires the problem to be completely reworked and no use is made of the previous estimate of β_j. This seems to waste effort. A sequential form for the estimate can be determined so that new observations can be incorporated without completely reworking the problem. The details are found in Curry and Bras (1980); the final forms of the equations follows.

The $(i+1)$th estimate of the coefficient vector β is given by

$$\beta_j^{i+1} = \beta_j^i + K_j^{i+1}\left(Z_j(i+1) - X_j(i+1)\beta_j^i\right). \qquad (3.123)$$

The vector K_j^{i+1} takes the form

$$K_j^{i+1} = \frac{S_j^{i+1}(X_j(i+1))^T}{\Psi_j(\mathrm{mod}_{12}(i+1))^2}. \qquad (3.124)$$

The variance of estimation is updated by

$$S_j^{i+1} = S_j^i - \frac{S_j^i(X_j(i+1))^T X_j(i+1)S_j^i}{\left[\Psi_j(\mathrm{mod}_{12}(i+1))^2 + X_j(i+1)S_j^i(X_j(i+1))^T\right]}. \qquad (3.125)$$

Two methods may be used to assign initial values to $\hat{\beta}_j$ and S_j. In the first (Panuska, 1969)

$$\beta_j^1 = \text{element of } B_j \qquad \forall j,$$

where B_j is a set known to contain β_j, and

$$S_j^1 = I \qquad \forall j.$$

The second method is

$$S_j^1 = \left[(X_j^1)^T(\Omega_j^1)^{-1}X_j^1\right]^{-1} \qquad \forall j$$

$$\beta_j^1 = S_j^1(X_j^1)^T(\Omega^1)^{-1}Z_j^1 \qquad \forall j.$$

The matrices X^1 are constructed using the available observations $Y_j(\cdot)$, $j = 1, \ldots, n_0$ and the indicated deterministic functions. The residuals $\{\varepsilon_j(\cdot), j = 1, 2, \ldots, n_0\}$ are replaced by randomly generated Gaussian deviates with zero mean and variance equal to $\Psi_j(\ell)^2$ (where $\ell = \mathrm{mod}_{12}(t)$), if available and

otherwise the sample variance of the observations, $Y_j(\cdot)$, is used. If the functions $\{\Psi_j(\ell), \forall j, \ell\}$ are not available, the Ω_j^1 is assumed to be $\mathbf{I}$ for all j.

The above estimation procedure deals with parameters of a single-station equation, independently of the others. Not accounting for the cross-correlated error between equations leads to inefficiency in the estimation. As in the algorithm defined on page 126, asymptotically efficient estimates of β may be obtained by extension to the simultaneous estimation of $\{\beta_j, j = 1, 2, \ldots, n_0\}$ rather than the estimation of each β_j separately. However, the computational burden (memory and execution time) would be greatly increased and the algorithm will not be recursive.

3.4.3 Tests of Model Adequacy

Chapter 2 emphasized that model building consists of repeatedly exercising identification, estimation, and verification steps. Verification of the generalized multivariate model follows the pattern of their univariate cousins.

Testing for parameter significance in the model with nonautocorrelated noise follows techniques common to OLS or linear regression. The availability of many books on that subject makes a detailed discussion here superfluous. (The reader is referred to Johnston [1972] and Draper and Smith [1966].) In estimating parameters for the nonautocorrelated-residuals model, it is recommended that a stepwise regression algorithm be used. This algorithm uses a F-test in selecting explanatory variables that yield significant coefficients according to the user's criteria.

The partial F-test and stepwise regression is not applicable to the model with autocorrelated noise, particularly since the explanatory variables include past residuals. The parameter error covariance matrix, S_j^N (Eq. 3.125) may be used to define reasonable ranges for the parameters. As a rule of thumb, parameters within the range $[-S_j^{1/2}(i,i), S_j^{1/2}(i,i)]$ may be considered insignificant. Another aid in verification is to study the evolution of parameter estimates. Parameters that exhibit wildly fluctuating behavior as new data points are processed imply that the model structure or the corresponding explanatory variable is not useful.

The properties of residuals are the best clue to model adequacy. By construction, the residual terms $V(k, i)$, both of models with correlated and models with uncorrelated residuals, should have zero mean and zero autocorrelations and cross-correlations at lags larger than one. Ideally the distribution of estimated residuals (obtained from the difference of observed and predicted values) should also be Gaussian. A few possible tests for residuals follow.

Durbin–Watson Test

Durbin and Watson (1950, 1951) derived a widely used test for autocorrelation of residuals. Although the test was derived for nonstochastic regression vari-

ables, it is applicable to a model with stochastic regression variables so long as they are serially independent and do not contain lagged values of the regressand or lagged noise terms (i.e., no autoregressive or moving-average terms). This being the case, the application of the test is only theoretically appropriate for the residuals of the model with nonautocorrelated residuals.[*]

The Durbin–Watson test is applied to the OLS residuals of each equation j of the monthly multivariate model with nonautocorrelated residuals by calculating the d statistic as

$$d_{ji} = \sum_{k=1}^{N/12} \left(\hat{V}_j(k,i) - \hat{V}_j(k-1,i) \right)^2 \Big/ \sum_{k=0}^{N/12} \hat{V}_j(k,i)^2. \qquad (3.126)$$

Upper (d_U) and lower (d_L) limits, dependent on the number of explanatory variables, observations $(N/12 + 1)$, and chosen significance level, are used to test the hypothesis of zero autocorrelation against the alternative hypothesis of positive first-order autocorrelation.

If $d_{ji} < d_L$, reject the hypothesis of nonautocorrelated residuals in favor of the hypothesis of positive autocorrelation.

If $d_{ji} > d_U$, do not reject the null hypothesis.

If $d_L < d_{ji} < d_U$, the test is inconclusive.

Values for d_L and d_U, tabulated against their arguments, may be found in Durbin and Watson (1951) and Johnston (1972).

To test the alternative hypothesis of negative first-order autocorrelation, compute $(4 - d_{ji})$ and compare this value to d_L and d_U as if testing for positive autocorrelation.

The above test is not appropriate for the residuals obtained from the GLS estimation of each multivariate monthly model with nonautocorrelated residuals. However, if these GLS residuals are grouped according to j and the Durbin–Watson statistic is calculated as indicated in Eq. (3.126), so that each monthly multivariate equation yields n_0 statistics, the test may be assumed to be approximately correct.

Durbin (1970) suggested a large-sample $(N > 30)$ procedure to test for autocorrelated residuals obtained from a model whose explanatory variables contain some autoregressive terms but no lagged noise (i.e., no moving-average terms).

Thus, the application of the test to the residuals of the model with autocorrelated noise may be assumed approximately correct when $n_1 \geq 1$ and

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[*]Assuming $n_1(i)$ is less than 12 for all i, the regression variables of nonautocorrelated noise models are not lagged values of the regressand, since they are different months. If $n_1(i)$ is greater than or equal to 12, the test may not be applicable theoretically. Serial independence of the regression variables is assumed approximately to hold, since they are separated by a lag of 12 months.

$n_2 = 0$, and when estimated by either least squares or weighted least squares. (Also, the test may be applicable to some residuals of the nonautocorrelated noise model if $n_1(i) \geq 12$ for any i.)

The test is applied to each equation j of the multivariate model (Eq. 3.104) by computing the statistics h_j, where

$$h_j = \frac{\sum\limits_{k=0}^{N/12} \sum\limits_{i=0}^{11} \hat{V}_j(k, i+1)\hat{V}_j(k, i)}{\sum\limits_{k=0}^{N/12} \sum\limits_{i=0}^{11} \hat{V}_j(k, i+1)^2} \sqrt{\frac{N}{1 - N \operatorname{var}(\hat{\beta}_j(1))}} \qquad (3.127)$$

and $\operatorname{var}(\hat{\beta}_j(1))$ equals the variance for the estimated coefficient of the first autoregressive term ($Y_j(k, i-1)$) (see Eq. 3.125).

Under the hypothesis of nonautocorrelated noise, the statistic h_j is distributed as a standard normal deviate. Thus, if $|h_j| > 1.96$, the hypothesis of zero autocorrelation is rejected at the 5% level of significance and if $|h_j| > 2.58$, the hypothesis is rejected at the 1% level of significance. Note that the test breaks down if $N \operatorname{var}(\hat{\beta}_j(1))$ is greater than one.

The tests of correlation of residuals discussed in Chapter 2 are also valid guidelines in the multivariate case.

Tests of Normality

The partial F-test used in the previously mentioned stepwise-regression procedure depends on the assumption of normally distributed residuals. As discussed in Chapter 2, normality can be tested for by studying skewness and kurtosis properties as well as χ^2 tests or visual inspection of residuals.

A coordinate system may be defined such that the cumulative distribution function (CDF) of any normal density function plots linearly. If the estimated CDF of a set of residuals plots approximately linearly in this coordinate system, it is a good indication that the residuals are normally distributed. Paper with this special coordinate system predefined may be obtained and is generally referred to as normal-probability paper.

The estimated CDF of a set of residuals $\{\hat{V}_j(k, i), \ k = 0, 1, 2, \ldots, N/12\}$ can be found. First, the residuals are ordered with respect to magnitude so that $\hat{V}_j^m(k, i)$ denotes the mth smallest value. Values of the CDF are then estimated by

$$\operatorname{prob}\left(V_j(\cdot) \leq \hat{V}_j^m(k, i)\right) \approx \frac{m}{(N/12)+1}.$$

Performance Evaluation

Many times, the ultimate use of the stochastic model is in the multi-lead forecast of monthly discharge. Thus, it is prudent to test the forecasting ability of a few candidate models before making a selection.

Let $\hat{Y}_j(k, i | k, i - \ell)$ denote a forecast of discharge at station j, year k, and month i, with a lead of ℓ months. The bias (B) and mean square error (MSE) of the forecasts may be tabulated over ℓ, i, and j, using the following relationships

$$B(\ell, i, j) = \frac{1}{(N/12)} \sum_{k=0}^{N/12} \left[Y_j(k, i) - \hat{Y}_j(k, i | k, i - \ell)\right] \qquad \forall j, i, \ell;$$

$$\qquad (3.128)$$

$$\operatorname{MSE}(\ell, i, j) = \frac{1}{(N/12)} \sum_{k=0}^{N/12} \left[Y_j(k, i) - \hat{Y}_j(k, i | k, i - \ell)\right]^2 \qquad \forall j, i, \ell.$$

$$\qquad (3.129)$$

Using the historical standard deviation of the sequence $\{Y_j(k, i), \ k = 0, 1, \ldots, N/12\}$, denoted by $S^2(j, i)$, a scaled statistic $R^2(\ell, i, j)$ may be computed from $\operatorname{MSE}(\ell, i, j)$ as

$$R^2(\ell, i, j) = \frac{S^2(j, i) - \operatorname{MSE}(\ell, i, j)}{S^2(j, i)}, \qquad (3.130)$$

where

$$S^2(j, i) = \frac{1}{(N/12)} \sum_{k=0}^{N/12} \left(Y_j(k, i) - \bar{Y}_j(i)\right)^2$$

and

$$\bar{Y}_j(i) = \frac{1}{(N/12)+1} \sum_{k=0}^{N/12} Y_j(k, i).$$

The statistics $B(\ell, i, j)$ and $R^2(\ell, i, j)$, $\forall i, j$, and $\ell = 1, 2, 3, \ldots$, are useful in evaluating the forecasting performance of a given model. Clearly, an unbiased forecast is desirable so that the closer the various values of B are to zero, the better the performance. Similarly, it is desirable to minimize the mean square error of forecasts and, thus, values of $R^2(\ell, i, j)$ close to 1 are desired.

3.4.4 An Example: Forecasting Flows in the Nile River

Curry and Bras (1980) evaluated several alternative forms of the models discussed in the previous section. The goal was to predict discharges in the Nile River, Egypt, and ultimately to use monthly forecasted flows to control releases from the High Aswan Dam (Bras et al., 1983). The most successful of the models corresponds to Eq. (3.103), a multivariate model with nonautocorrelated residuals.

The model was calibrated using monthly data from 1912 to 1965 inclusive, in all eight stations schematized in Fig. 3.16. Models for all eight stations were evaluated simultaneously using the algorithm discussed on pages 126–130. For the sake of expediency only results of the equation for Wadi Halfa, at the entrance to Lake Nasser, are given.

Table 3.1 gives the selected explanatory variables and estimated coefficients for all 12 months in Wadi Halfa. For comparison, both Ordinary Least Squares (OLS) and Generalized Least Squares (GLS) results are given. Of interest is the different nature of each monthly equation; each has very different explanatory variables and coefficients.

Table 3.2 gives the results of a Durbin–Watson test on residuals. The hypothesis of uncorrelated residuals could not be rejected in the great majority of the equations. In some the test was inconclusive or not applicable. In four cases, the hypothesis was rejected.

The calibrated model was used for forecasting at various leads. Forecasting is accomplished using the expected value of Eq. (3.103). For example, a forecast one month ahead (lead one) is

$$\hat{\mathbf{Y}}(k, i) = \hat{\boldsymbol{\Phi}}_i \begin{bmatrix} \mathbf{Y}(k, i-1) \\ \mathbf{Y}(k, i-2) \\ \vdots \\ \mathbf{Y}(k, i - n_1(i)) \end{bmatrix} + \hat{\mathbf{U}}(i), \qquad (3.131)$$

where the symbol "^" designates an estimate. At lead one all explanatory variables are actual observations. If multi-lead forecasts are required, Eq. (3.131) is used recursively. For example, a lead-two forecast for month i (from the end of month $i - 2$) utilizes the lead-one forecasting equation for month i, replacing lag-one explanatory variables (which have not been observed at the time of the lead-two forecast) with the lead-one predictions generated from the end of month $i - 2$. Using both the lead-one and lead-two predictions, as well as all observations prior to and including month $i - 2$, lead-three forecasts can be generated, and so forth. Further details of the forecasting procedures for all models are found in Curry and Bras (1980).

The resulting R^2 statistics of predicting all months at various lags were computed using Eq. (3.130) and are given in Table 3.3. Considerable variation exists in the accuracy of forecasts for any given lead. For example, the dry month of April at Wadi Halfa can be forecast in March (lead 1) with a variance reduction of 96%, while the 12-month lead forecast has a variance reduction of 58%. On the other hand, the flood month of August at Wadi Halfa is forecast with a 75% variance reduction at lead 1, but only a 1% variance reduction with a six-month lead.

Tables 3.4 through 3.6 (Bras et al., 1983) give examples of actual predictions and comparisons to observed discharges. Forecasts are given for three years from the end of January, April, and August, for various leads. For example, from the end of January 1922, the model predicts reasonably well
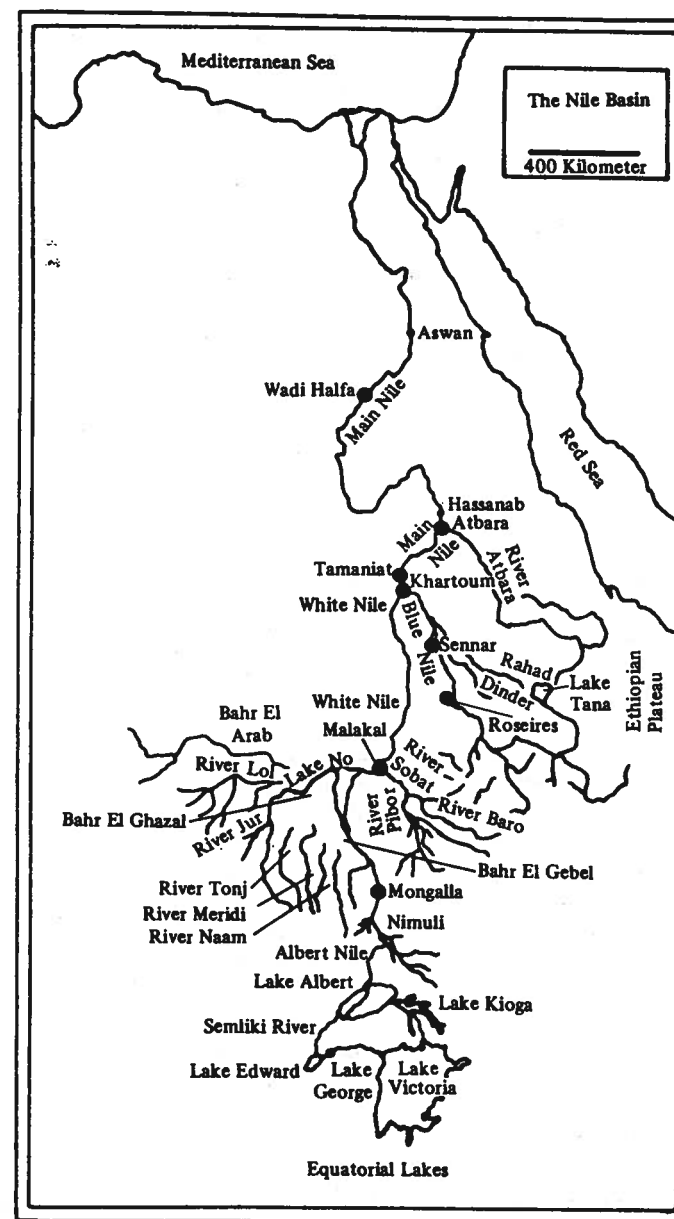
Figure 3.16 Map of Nile River Basin.

Table 3.6
Forecasts and actual observations from the end of August for the years 1913, 1941, and 1964 in 10^9 m^3

Lead	Forecasts, observations						Forecast	Sample statistics	
	1913		1941		1964				
	Forecast	Observation	Forecast	Observation	Forecast	Observation	Standard deviation	Mean	Standard deviation
1	14.92	13.40	17.57	15.10	24.52	25.70	2.01	21.99	3.78
2	10.72	7.86	13.34	11.10	16.10	16.40	2.39	14.61	3.15
3	6.11	4.14	6.26	7.43	7.47	9.45	1.57	7.17	1.77
4	3.83	3.02	3.92	3.99	6.00	5.85	0.64	4.54	0.87
5	2.08	2.10	2.10	2.99	4.05	5.75	0.44	3.51	0.74
6	1.57	1.42	1.59	2.08	3.89	5.08	0.42	2.45	0.69
7	2.02	1.29	2.04	2.54	3.90	4.39	0.48	2.28	0.67
8	1.42	1.10	1.92	2.52	3.32	3.58	0.38	2.04	0.69
9	1.33	1.18	1.56	1.81	3.50	4.17	0.45	1.92	0.76
10	1.75	1.14	1.78	1.92	3.53	4.52	0.60	2.07	0.78

Bras et al., 1983.

(up through July) the below-average flows. Similarly, the 1965 high flows are well predicted from the end of January. Note that as the lead increases, the forecast deteriorates and approaches the historical mean. On June 1965 the observed flow was 4.52×10^9 m^3. At lead two, from the end of April, this month was forecast as 4.54×10^9 m^3. At lead 5, from the end of January, the forecast was 4.17×10^9 m^3, still very good. From the end of August 1964 (lead 10) the forecast is 3.53×10^9 m^3. This quantity has a much higher error; nevertheless, the model still recognizes flows well above the 2.07×10^9 m^3 historical-mean.

3.5 THE DISAGGREGATION MODEL

Section 2.5.2 discussed seasonal univariate models. It should be evident by now that model building to preserve multi-lag correlations among the various seasons can be fairly complicated and tedious. Failure to represent these correlations adequately can lead to serious errors in analysis. For example, reservoirs may be underdesigned if seasonal correlations at lags higher than one are significant and they are not taken into account. Blind use of the seasonal lag-one (Thomas–Fiering) model does indeed commonly result in this type of error. Valencia and Schaake (1972, 1973) suggested preserving all interseasonal correlations by appropriately disaggregating annual (aggregated) variables. Throughout this section, then, assume that a set of aggregated random variables is available (i.e., annual streamflow at n locations) and it is desired to obtain a corresponding series of seasonal variables (i.e., monthly streamflows at all n locations). This disaggregation must be performed so that all correlations among seasons (months) and between variables (sites) will be preserved. Logically, seasonal values must add up to the original aggregated variables.

Assume you have a vector of annual values of discharge at different locations:

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad (3.132)$$

where n is the number of sites. Define a vector of seasonal (monthly, quarterly, etc.) values of discharge at the n sites

$$Y = \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{m1} \\ \vdots \\ y_{1n} \\ \vdots \\ y_{mn} \end{bmatrix}, \qquad (3.133)$$

where y_{ij} is the discharge at location j during season i. There are m seasons.

Note that

$$x_j = \sum_{i=1}^{m} y_{ij} \qquad (3.134)$$

or

$$X = CY,$$

where

$$C = \begin{bmatrix} 1\cdots1 & 0\cdots0 & \cdots & 0\cdots0 \\ 0\cdots0 & 1\cdots1 & \cdots & 0\cdots0 \\ \vdots & & & \\ 0\cdots & \cdots & \cdots & 1\cdots1 \\ & & & \underleftrightarrow{\quad} \\ & & & m \end{bmatrix} \qquad (3.135)$$

The disaggregation model takes the form

$$Y = AX + BW, \qquad (3.136)$$

where Y and X are zero mean vectors (mean subtracted), A is an $nm \times n$ matrix, B is an $nm \times nm$ matrix, W is an $nm \times 1$ vector of standard normal deviates, and $E[WW^T] = I$, $E[BWY^T] = 0$.

Parameters A and B will be selected to preserve the correct cumulative relation (Eq. 3.134), the spatial correlation between all stations and months, and the correlation between the aggregated (X) and disaggregated (Y) vectors.

The above equation is clearly analogous to the expression for the multivariate AR(1) model (Eq. 3.1). The parameter matrices A and B are then obtained according to equations equivalent to Eqs. (3.7) and (3.8)

$$A = S_{yx} S_{xx}^{-1} \qquad (3.137)$$

$$BB^T = S_{yy} - S_{yx} S_{xx}^{-1} S_{xy}, \qquad (3.138)$$

where $S_{xx} = E[XX^T]$, $S_{yy} = E[YY^T]$, and $S_{yx} = E[YX^T]$. The resulting BB^T matrix is always positive semidefinite, with rank $nm - n$. The rank is not full because there must be n dependent columns in the matrix. This results from the fact the sum of the seasons must add up to the annual values, leaving only $m - 1$ degrees of freedom per station. The decomposition of BB^T is then usually done using the principal-components method discussed on pages 95–96.

Sample estimates of S_{yx}, S_{xx}, and S_{yy} are obtained with equations equivalent to Eqs. (3.12) and (3.13).

As long as A and B are obtained as previously described, the disaggregation model will have the following properties:

1. Preserve mean of vector Y
2. Preserve variance of elements of Y
3. Preserve the cross-correlation matrix between Y and X, S_{yx}
4. Preserve the correlation among the elements of Y, S_{yy}
5. Preserve the cumulative relation, $X = CY$, therefore, the disaggregated values add up to the annual components.

The last property is easily shown. Equation (3.136) will satisfy any linear relation

$$X = CY. \qquad (3.139)$$

Using Eq. (3.139)

$$S_{yx} = S_{yy} C^T$$
$$S_{xx} = E[XX^T] = C S_{yy} C^T \qquad (3.140)$$
$$S_{xy} = C S_{yy}.$$

Substituting into the expressions for A and BB^T (Eqs. 3.137 and 3.138),

$$A = S_{yy} C^T (C S_{yy} C^T)^{-1} \qquad (3.141)$$

$$BB^T = S_{yy} - S_{yy} C^T (C S_{yy} C^T)^{-1} C S_{yy}. \qquad (3.142)$$

Pre-multiplying A by C leads to

$$CA = C S_{yy} C^T (C S_{yy} C^T)^{-1} = I. \qquad (3.143)$$

Pre-multiplying BB^T by C yields

$$CBB^T = C S_{yy} - C S_{yy} C^T (C S_{yy} C^T)^{-1} C S_{yy} = 0. \qquad (3.144)$$

Eq. (3.144) implies

$$CB = 0. \qquad (3.145)$$

It is then easy, using Eqs. (3.143) and (3.145), to premultiply Eq. (3.136) by C, leading to

$$CY = CAX + CBW$$
$$= X,$$

which is the desired linear relation.

The conditions

$$CA = I$$
$$CB = 0$$

are good checks for the correct estimation of A and B.

3.5.1 Comments and an Example on Disaggregation

The disaggregation model is possibly one of the most widely accepted tools in stochastic hydrology. Conceptually it is simple, but nevertheless it accomplishes a considerable amount of otherwise very difficult tasks. In practice, the model can lead to various numerical problems. Essentially, the decomposition of the matrix BB^T can rapidly deteriorate numerically, particularly if its dimensions ($mn \times mn$) are large. Similarly, the necessary inversion of matrix S_{xx} can lead to numerical errors. Computer storage can also be a problem. Note that in a not uncommon problem of 6 stations and 12 seasons, the Y vector consists of 72 elements and the largest matrix will be of dimensions 72×72, a considerable storage demand. For accuracy, double precision is always recommended. In order to save space, it is possible to disaggregate in various stages, as shown in Fig. 3.17. Note, for example, that quarterly values are disaggregated into months independently. In such a case the correlation of the last month of every quarter and the first month of the next quarter will not be explicitly preserved. This behavior is shown in Fig. 3.18. Shown are lag-one correlations in four streamflow stations in the Nile River, Egypt. Compared to the historical values are those obtained by synthetic generation with a seasonal multivariate AR(1) model (Section 3.3.1) and by disaggregation of annual values generated by a multivariate broken-line model, to be described in Chapter 5. The nature of the approximation schematized in Fig. 3.17 is obvious. The disaggregated values fail to preserve the correlation between adjacent months in different seasons. Nevertheless, since correlations between and within quarters will be correctly handled, the errors incurred should be minimal. Alternatively, the simple step-disaggregation proposed by Santos and Salas (1983) may be useful here.

The reader should be able to recognize that the general disaggregation scheme also ignores seasonal correlations between different years, in particular the correlation between the last and first seasons of adjacent years. If necessary, this can be completely solved by adding additional elements to vector Y (state augmentation) or, more simply, by modifying Eq. (3.136) to include an additional linear dependency between vector Y and seasons corresponding to the previous year (previous X value). Mejia and Rouselle (1976) discuss such an approach in detail, although their particular approach is flawed (Valencia et al., 1983).

The disaggregation model operates on aggregated values, regardless of their origin. Typical applications would be the generation of seasonal stream-
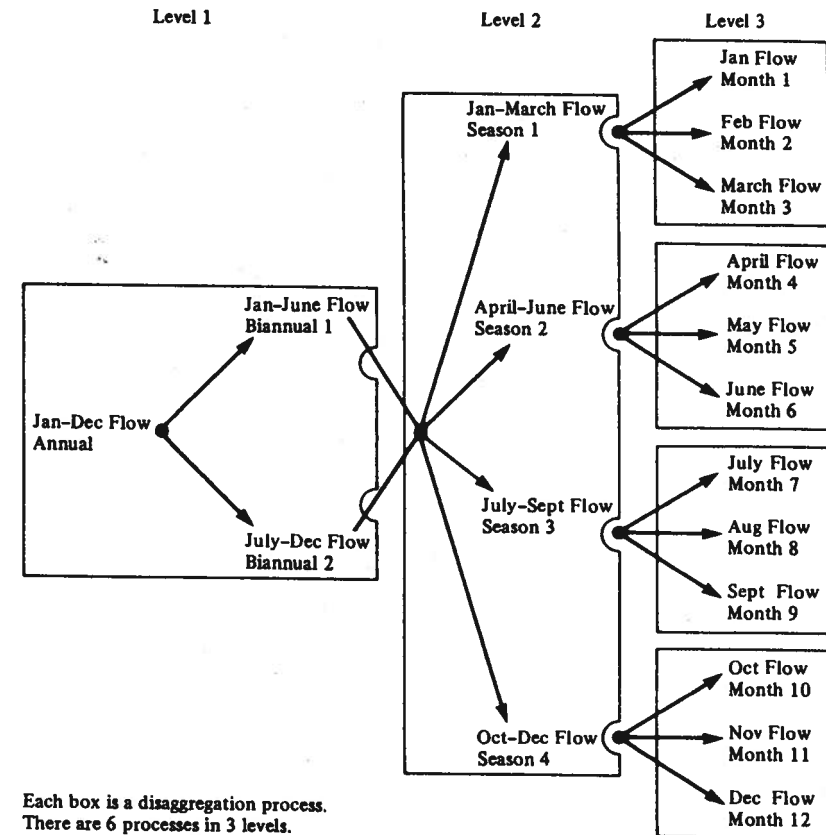
Each box is a disaggregation process.
There are 6 processes in 3 levels.

Figure 3.17 Disaggregation scheme in various stages (from Curry and Bras, 1978 [with the permission of Kevin Curry, Sperry Flight Systems, Phoenix, Ariz.]).

flows from an available annual record; or the disaggregation of annual streamflows generated using another time-series model. (Chapter 2 discussed univariate models capable of annual streamflow generation.)

The multivariate seasonal AR(1) (mixed distributions) proved to be a very good model of the Nile River. The correlation structure at lags higher than the explicitly preserved lag-one was reasonably well reproduced, at least down to the quarterly level. Even then, the performance of the AR(1) in preserving statistics at aggregated levels, such as the annual level, is not the best. For example, Table 3.7 gives historical lag-one correlations of annual flows in four stations in the Nile. Given for comparison are the annual lag-one correlations resulting from the broken-line model, which are unaltered by disaggregation,
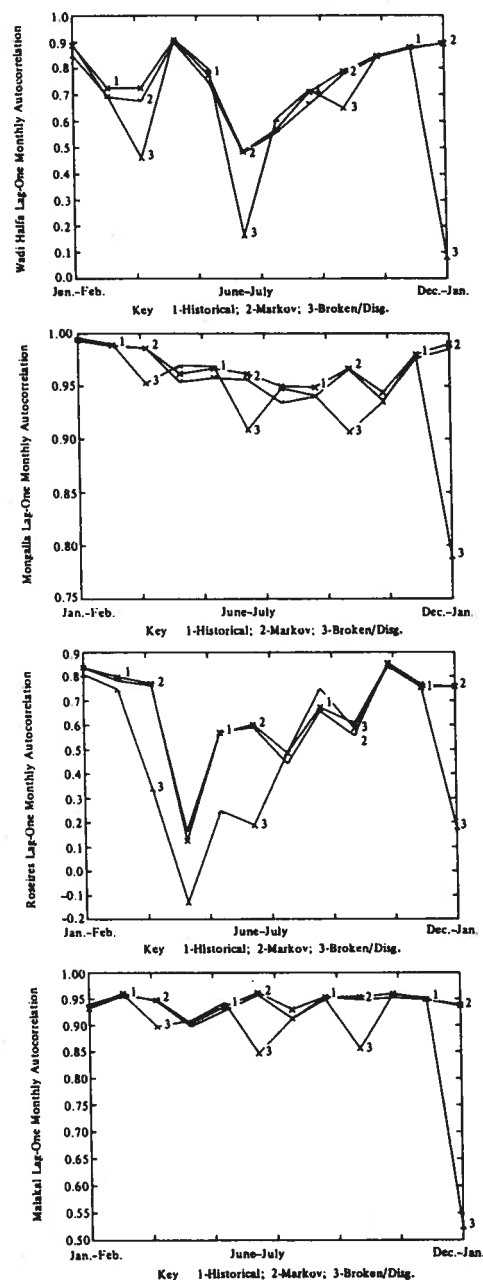
Figure 3.18 Lag-one monthly autocorrelations for the Nile River at Wadi Halfa, Mongalla, Roseires, and Malakal: (1) historical values, (2) values from a seasonal multivariate AR(1) model, (3) values from the disaggregation of annual flows generated by a multivariate broken-line model (from Curry and Bras, 1978 [with the permission of Kevin Curry, Sperry Flight Systems, Phoenix, Ariz.]).

TABLE 3.7

Comparison of annual Nile River lag-one correlations as obtained by the multivariate seasonal AR(1) model and the broken-line model

	Lag-one autocorrelations of annual streamflow		
	Historical	*AR(1)*	*Broken line*
Wadi Halfa	0.184	0.089	0.207
Mongalla	0.886	0.825	0.881
Roseires	0.162	0.067	0.146
Malakal	0.801	0.752	0.786

Curry and Bras, 1978 (with the permission of Kevin Curry, Sperry Flight Systems, Phoenix, Ariz.).

Table 3.8

Storage required* to meet monthly target releases where annual target is 70% of mean annual flow[†]

40-year reliability (%)	Thomas–Fiering monthly flow model	AR(1) annual flow generator	ARMA(1,1) annual flow generator
80	0.31	0.37	0.38
95	0.34	0.50	0.61
98	0.36	0.62	0.83

*Storage reported as a fraction of mean annual flow.
[†] Values for Ishikari River, Japan, based on study reported in Hoshi et al. (1978).

Louks, D. P. et al., 1981, *Water Resource Systems Planning and Analysis*, Englewood Cliffs, N.J.: Prentice-Hall).

and those resulting from the aggregated monthly streamflows generated by the seasonal AR(1). Even in this favorable example the AR(1) is clearly inferior in this respect.

The possible value of the disaggregation model is well illustrated by Hoshi et al. (1978), as reviewed by Loucks et al. (1981). Table 3.8 summarizes a storage-yield analysis performed with three different models: a monthly lag-one autoregressive model (Section 2.5.2, Eq. 2.103), an AR(1), and an ARMA(1,1). The last two nonseasonal models were then coupled with a disaggregation model to obtain monthly streamflows. For high-release targets, such as the 70% of mean annual flow shown, the disaggregation-based procedures always yielded larger reservoirs. The required reservoir size increased significantly with

reliability for both the AR(1) and ARMA(1,1) models. This behavior is what motivated Valencia and Schaake to formulate the disaggregation model. Monthly correlations at large lags can be significant enough to affect reservoir design and operation.

Finally, you should realize that the disaggregation model is not parsimonious. It requires a lot of parameters and data. Improvements requiring less parameters have been presented by Santos and Salas (1983), Stedinger (1983), and Lane (1983).

CHAPTER 4

Frequency-Domain Analysis of Hydrologic Processes

4.1 INTRODUCTION

In the past three chapters we have discussed random processes in the time domain. All properties, such as correlation, were studied in terms of the original argument of the process, or space of definition, generally taken as time in hydrology. Most people prefer to think in terms of the daily concept of time, but the fact is that transformation of the process domain to frequency can yield considerable insight. Historically the analysis of signals, including random signals, was considerably advanced in the frequency domain. The works of Wiener (1949) and his student Lee (1960), dealing with transformed processes defined in the frequency domain, form the basis of the material in this chapter. Emphasis on time-domain analysis is relatively recent, probably because of the availability and popularity of numerical, computer-based solutions to differential equations, eliminating the need for the analytic integrals in frequency-domain work. This newly discovered freedom has also led to advances and solutions that were stifled by the old approach.

This chapter will familiarize the reader with the traditional concepts of frequency-domain analysis. We intend not only to provide the right historical