5. DUAL LP, SOLUTION INTERPRETATION, AND POST-OPTIMALITY

5.1 DUALITY

Associated with every linear programming problem (the primal) is another linear programming problem called its dual. If the primal involves \( n \) variables and \( m \) constraints, the dual involves \( n \) constraints and \( m \) variables. The solution to either is sufficient for readily obtaining the solution to the other. In fact, it is immaterial which is designated as the primal since the dual of a dual is the primal.

Suppose that the primal problem is:

Max \( Z = C_1 X_1 + C_2 X_2 + \ldots + C_n X_n \) \[5.1\]

s.t.: \[\begin{align*}
A_{11} X_1 + A_{12} X_2 &+ \ldots + A_{1n} X_n \leq B_1 \\
A_{21} X_1 + A_{22} X_2 &+ \ldots + A_{2n} X_n \leq B_2 \\
& \vdots \\
A_{m1} X_1 + A_{m2} X_2 &+ \ldots + A_{mn} X_n \leq B_m \\
X_j \geq 0 & \quad \text{(for } j = 1, 2, \ldots, n) 
\end{align*}\] \[5.2-5.3\]

or

Max \( Z = C^T X \) \[5.4\]

s.t.: \( AX \leq B \) \[5.5\]

If the above (Expressions [5.1] through [5.5]) is the primal LP problem, the dual problem is:

Min \( \tilde{Z} = B_1 Y_1 + B_2 Y_2 + \ldots + B_m Y_m \) \[5.6\]

s.t.: \[\begin{align*}
A_{11} Y_1 + A_{21} Y_2 &+ \ldots + A_{m1} Y_m \geq C_1 \\
A_{12} Y_1 + A_{22} Y_2 &+ \ldots + A_{m2} Y_m \geq C_2 \\
& \vdots \\
A_{1n} Y_1 + A_{2n} Y_2 &+ \ldots + A_{mn} Y_m \geq C_n 
\end{align*}\] \[5.7-5.9\]
\[ Y_i \geq 0 \quad \text{for all } i = 1, 2, \ldots, m \quad \ldots[5.10] \]

or

\[
\min \tilde{Z} = B^T Y 
\]

s.t.: \[ A^T Y \geq C \quad \ldots[5.12] \]

The variables \( Y_i \) are called the “dual variables”. One dual variable will be associated with each constraint in the primal problem. Note that equality constraints are either replaced by two inequalities or, if left as an equality, then the associated dual variable is unrestricted in sign.

5.2 AN EXAMPLE OF THE PRIMAL-DUAL RELATIONSHIP

Consider the following primal problem and its corresponding dual:

**Primal Problem:**

\[
\min Z_x = 10 X_1 + 4 X_2 
\]

**Dual Variables**

s.t.: \[ X_1 \geq 4 \quad Y_1 \quad \ldots[5.14] \]

\[ X_2 \geq 6 \quad Y_2 \quad \ldots[5.15] \]

\[ X_1 + 2X_2 \geq 20 \quad Y_3 \quad \ldots[5.16] \]

\[ 2X_1 + X_2 \geq 17 \quad Y_4 \quad \ldots[5.17] \]

**Dual Problem:**

\[
\max Z_y = 4 Y_1 + 6 Y_2 + 20 Y_3 + 17 Y_4 
\]

s.t.: \[ Y_1 + Y_3 + 2Y_4 \leq 10 \quad \ldots[5.19] \]

\[ Y_2 + 2Y_3 + Y_4 \leq 4 \quad \ldots[5.20] \]

The solutions for these two linear optimization problems are:
<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 = 4$</td>
<td>$Y_1 = 2$</td>
</tr>
<tr>
<td>$X_2 = 9$</td>
<td>$Y_2 = 0$</td>
</tr>
<tr>
<td>$S_1 = 0$</td>
<td>$Y_3 = 0$</td>
</tr>
<tr>
<td>$S_2 = 3$</td>
<td>$Y_4 = 4$</td>
</tr>
<tr>
<td>$S_3 = 2$</td>
<td>$S_1 = 0$</td>
</tr>
<tr>
<td>$S_4 = 0$</td>
<td>$S_2 = 0$</td>
</tr>
<tr>
<td>$Z = 76$</td>
<td>$\tilde{Z} = 76$</td>
</tr>
</tbody>
</table>

Note that the dual is rather more easily solved using the simplex method because of the existence of a basic feasible solution without resorting to artificial variables, and also having two rather than four constraints.

The optimal value of the non-zero portions of the objective functions for both forms of the example problem are:

\[ Z_y = 4 (2) + 17 (4) = 76 \quad \text{(dual)} \quad \text{...[5.21]} \]

and

\[ Z_x = 10 (4) + 4 (9) = 76 \quad \text{(primal)} \quad \text{...[5.21]} \]

A more complete display of information provided by solutions to both forms of the example problem is given on the two LINGO solutions in Table 5.1. Note that the dual prices of the primal solution are the negatives of the values of the dual solution decision variables, and the dual prices of the dual solution are equal to the values of the decision variables in the primal solution.

The Dual Theorem can be stated as:

Let:

\[ Z^* = \sum_{j=1}^{n} c_j x^*_j \quad \text{...[5.22]} \]

and

\[ Z^*_y = \sum_{i=1}^{m} b_i y^*_i \quad \text{...[5.23]} \]
Table 5.1: Comparison of Primal and Dual Problem Solutions (obtained with LINGO)

<table>
<thead>
<tr>
<th>Model Type</th>
<th>LINGO Model Formulation*</th>
<th>LINGO Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primal</td>
<td>model: min = 10 * x1 + 4 * x2; x1 - s1 = 4; x2 - s2 = 6; x1 + 2 * x2 - s3 = 20; 2 * x1 + x2 - s4 = 17; end</td>
<td>Global optimal solution found at step: 4 Objective value: 76.00000</td>
</tr>
<tr>
<td></td>
<td>Variable</td>
<td>Value</td>
</tr>
<tr>
<td></td>
<td>X1</td>
<td>4.000000</td>
</tr>
<tr>
<td></td>
<td>X2</td>
<td>9.000000</td>
</tr>
<tr>
<td></td>
<td>S1</td>
<td>0.0000000</td>
</tr>
<tr>
<td></td>
<td>S2</td>
<td>3.000000</td>
</tr>
<tr>
<td></td>
<td>S3</td>
<td>2.000000</td>
</tr>
<tr>
<td></td>
<td>S4</td>
<td>0.0000000</td>
</tr>
<tr>
<td></td>
<td>Row</td>
<td>Slack or Surplus</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>76.00000</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.0000000</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.0000000</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.0000000</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.0000000</td>
</tr>
<tr>
<td>Dual</td>
<td>model: max = 4 * y1 + 6 * y2 + 20 * y3 + 17 * y4; y1 + y3 + 2 * y4 + s1 = 10; y2 + 2 * y3 + y4 + s2 = 4; end</td>
<td>Global optimal solution found at step: 4 Objective value: 76.00000</td>
</tr>
<tr>
<td></td>
<td>Variable</td>
<td>Value</td>
</tr>
<tr>
<td></td>
<td>Y1</td>
<td>2.000000</td>
</tr>
<tr>
<td></td>
<td>Y2</td>
<td>0.0000000</td>
</tr>
<tr>
<td></td>
<td>Y3</td>
<td>0.0000000</td>
</tr>
<tr>
<td></td>
<td>Y4</td>
<td>4.000000</td>
</tr>
<tr>
<td></td>
<td>S1</td>
<td>0.0000000</td>
</tr>
<tr>
<td></td>
<td>S2</td>
<td>0.0000000</td>
</tr>
<tr>
<td></td>
<td>Row</td>
<td>Slack or Surplus</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>76.00000</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.0000000</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.0000000</td>
</tr>
</tbody>
</table>

*Note: Inequality constraints have been replaced with equality constraints containing appropriate slack or surplus variables.

be the optimal values of objective function and solution variables. If a finite feasible solution exists for both the primal and dual problems, then there exists a finite optimal solution for both problems where

\[ Z^*_X = Z^*_Y \] \[ \text{...[5.24]} \]

In other words, the maximum feasible value of the primal objective function equals the minimum feasible value of the dual objective function. Note then for any feasible solution,

\[ Z_X \leq Z_Y \] \[ \text{...[5.25]} \]

Given the following definitions of variables and coefficients, an intuitive interpretation of the primal-dual relationship is:
Primal
Max $Z_X = C^T X$

s.t.: $AX \leq B$

$X = \text{vector of activities (levels of production)}$
$C = \text{vector of unit benefits ($/activity)}$
$B = \text{vector of resources available}$

Hence, $A = \text{resources/activity and } Z_X = (\text{benefit / activity}) \times \text{activity} = \text{benefit}$

Dual
Min $Z_Y = B^T Y$

s.t.: $A^T Y \geq C^T$

$Y = \text{vector of unit values of resources used}$

Also, $Y_i = \frac{\partial Z^*}{\partial b_i}$

$A^T Y \geq C^T$ is $(\text{resource / activity}) Y \geq \text{benefit / activity}$

Thus $Z_Y = \text{resource (benefit / resource)}$

$= \text{benefit}$

If the primal problem is to Maximize Benefits, then the dual is a cost minimization problem in which the value of scarce resources used in producing the benefits are minimized. Shadow prices (“dual variables”) $Y_i$ define the marginal value of the contribution of each constraint to the objective function and answers the question of how much more output (value) can be obtained by relaxing a constraint by one unit.

5.3 LP SOLUTION INTERPRETATIONS

In general for practical LP problems, the model parameters ($c_j$, $a_{ij}$, and $b_i$) are seldom known with certainty, or the parameter values or model structure may be subject to change over time. Thus, it is important to investigate the sensitivity of optimal solutions to change in parameter values. Two questions are of interest:

1. What are the ranges of parameter values for which the model solution remains optimal?

2. For new parameter values is the solution still optimal? What is the new optimal solution?

To begin to interpret the solution to an LP problem, one should determine which decision variables are basic (which $X_j$ are greater than zero) and which are non-basic; also, which constraints are binding (which constraints contain slack variables equal to zero), and which are non-binding. Why should this be important? For example, although we treat the limits on resources (the $B_i$ vector) as constants, these values are often only estimates of average or critical levels of stochastic variables, such as streamflow. If a particular variable is not basic, or if a constraint is not binding, then we are not so worried about the uncertainty related to our estimates of those resources. If on the other hand, a variable is active in the solution, or if a constraint is binding,
then we would like to know how important are errors in our estimates of those resources or of unit costs or benefits from those variables. Various types of sensitivity analysis can answer such questions.

A graphic interpretation of this type of sensitivity question is useful. Figure 5.1 displays an example of the effect of varying the limit of a resource that is binding. Note that constraints 2 and 3 are binding (and therefore have dual variables that are not zero), that variables $X_1$ and $X_2$ are basic, and the slack variables for constraints 2 and 3 are non-basic (i.e., are equal to zero). The optimal solution is shown at point B.

![Figure 5.1: Illustration of LP Solution Sensitivity to Uncertainty in Resource Availability](image)

Now assume that the amount of resource 1 (i.e., the right-hand-side of constraint 2) is reduced, causing the constraint to shift to position 2'. This causes $Z^*$ to reduce, $X_1$ to increase, and $X_2$ to decrease. However, the basic variables remain the same. The optimal solution is now B'. Because the basis has not changed, the value of one more (or less) unit of resource 2 has not changed--therefore the shadow price of resource 2 (which is the dual variable $Y_2$) has not changed. This means that $\frac{\partial Z^*}{\partial b_2} = Y_2 = b_2$ and $\frac{\partial Z^*}{\partial b_3} = Y_3 = b_3$ have not changed. If fact, these shadow prices, or imputed values of resources 2 and 3, will stay constant as the limit on resource 2 continues to decrease until constraint 2 passes through point C. Then a change of basis will occur, making constraint 3 become non-binding, and constraint 4 will become binding. This means that the dual variable $Y_3$ will now be zero, and $Y_2$ will be non-zero (but different than before, and $Y_4$ will be non-zero).
Some LP algorithms have a utility called RANGE that produces information on the range over which and right-hand-side (RHS) can vary before the shadow prices (the dual variables) change. The LINGO solutions give the value of the shadow prices in the column of the ROWS section labeled DUAL PRICE. We now have several names for the same parameter, as follows:

\[ Y_i = \bar{X}_i = \text{shadow price} = \text{imputed value of resource} = \text{dual price} = \text{dual variable} = \frac{\partial Z^*}{\partial b_i} \]

### 5.4 SENSITIVITY ANALYSIS OF OBJECTIVE COEFFICIENTS (\(C_j\))

Another type of sensitivity analysis considers the effect of varying the values of the cost, or \(C_j\), coefficients (the value or cost of one unit of \(X_j\)). Imagine what would happen if the values of either \(C_1\) or \(C_2\) or both were to change. This would cause the slope of the objective function, \(Z\), to increase or decrease. This would change the value of \(Z^*\), but not the optimal level of \(X_1\) or \(X_2\) (and therefore there would not be a basis change) unless the change causes the slope of \(Z\) to increase enough to equal or exceed the slope of constraint 3 (refer to Figure 5.1). At this point a change of basic variables would occur, and \(X_1^*\) and \(X_2^*\) would change.

The LINGO solution file has a column in the Variables section labeled REDUCED COST. This column gives the change in \(Z^*\) per unit \(X_j\) if a non-basic \(X_j\) were to enter the basis. That is,

\[ \text{REDUCED COST} = \frac{\partial Z^*}{\partial x_j} \]

for non-basic \(X_j\).

### 5.5 PARAMETRIC PROGRAMMING

The following distinction is made between sensitivity analysis and parametric programming:

1. **Sensitivity analysis**—examination of discrete parameter changes
2. **Parametric programming**—analysis of changes in optimal solutions for continuous or systematic parameter changes of one or several parameters simultaneously.

The general types of parameter and model structure changes to be examined in LP problems are:

1. Objective function coefficients, \(C_j\)
2. Resource limits or RHS values, \(B_i\)
3. Changes in technical (constraint) coefficients, $A_{ij}$

4. Effect of an additional constraint

5. Effect of additional variables

5.5.1 Changes in Objective Function Coefficients, $C_j$

To evaluate changes in objective function coefficients, the original objective function

$$Z = \sum_{j=1}^{n} C_j X_j$$  \hspace{1cm} \text{...[5.27]}$$

is rewritten as:

$$Z = \sum_{j=1}^{n} (C_j + Q_j) X_j$$  \hspace{1cm} \text{...[5.28]}$$

where the $Q_j$s are relative rates of change for the $C_j$s. Note, for $Q_j = 0$, we have the original LP problem. $Q_j$ varies from zero to some specified positive number, V, such that $0 \leq Q_j \leq V$. The goal is to find an optimal solution to the modified LP problem (subject to the original constraints) as a function of $Q_j$. The parametric analysis must determine when and how the optimal solution changes, and to what, over the range of $Q_j$.

5.5.2 Changes in Resource Limits, $B_i$

The effects of changes in resource limits as expressed in an LP model can be evaluated in a fashion similar to the above analysis of the $C_j$s. For this, the original constraint

$$\sum_{j=1}^{n} A_{ij} X_j \leq B_i \hspace{1cm} (i = 1, 2, \ldots, m) \hspace{1cm} \text{...[5.29]}$$

is rewritten as:

$$\sum_{j=1}^{n} A_{ij} X_j \leq B_i + Q_i \hspace{1cm} (i = 1, 2, \ldots, m) \hspace{1cm} \text{...[5.30]}$$

and the LP problem is then solved as the value of $Q_i$ is varied over a range. The goal is to identify changes in optimal solutions as a function of $Q_i$. 

72
5.5.3 **Strategy of Post-Optimality Analysis**

Clearly, adjusting the model, resolving, and comparing could get answers to questions posed. Research devoted to the subject has developed more efficient procedures by working from the present optimal solution. Procedures are usually available as part of standard computer LP packages.
5.6 PROBLEMS

1. Solve (using LINGO) both the primal and the dual of the following problem and compare the solutions:

Max $Z = 2A + 3B + C$

s.t.:

$A + 5B \leq 15$

$A + B + C \geq 6$

$A \geq B + C$

a. What are the dual prices of the primal? How do they compare to the decision variables of the dual?

b. What are the dual prices of the dual, and how do they compare with the decision variables of the primal?

2. Repeat Problem 1, above, with the inequality in the last constraint changed to an equality.

Note: The following procedure will help avoid errors in the homework, and is highly recommended:

a. Before writing the dual, always convert the constraints into the following form:

- for maximization problems, make all constraints take the form $g(x) \leq b$ by changing signs if necessary

- for minimization problems, make all constraints take the form $g(x) \geq b$, again by changing signs if necessary

b. For a strict equality constraint, specify that the corresponding dual variable is unrestricted in sign. However, since the simplex method does not in general allow negative decision variables, if you were doing a simplex solution by hand you could use the following transform:

$$Y_1 = Y_a - Y_b$$

Then, $Y_a$ and $Y_b$ would be set in the optimal simplex solution to be nonnegative.

For a manual derivation of the dual, replace the single primal constraint with two constraints that are it's equivalent. For example, if the primal problem includes:
Max Z

s.t.: \( 2X_1 + X_2 = 5 \)

replace the above equality constraint with the following two constraints:

\( 2X_1 + X_2 \leq 5 \)

\( 2X_1 + X_2 \geq 5 \)

Then convert the second inequality constraint into:

\( -2X_1 - X_2 \leq -5 \)

before writing the dual.