8. PIECEWISE LINEARIZATION

8.1 INTRODUCTION

Most water resource planning and/or operation problems can be expressed in terms of linear constraints. Mass balance or limits on resource use, for example, are generally linear functions. Many objective functions, however, tend to be non-linear. Design problems for which the objective is to minimize cost or maximize benefits minus costs usually have cost functions with economies of scale. This implies a non-linear function as shown in Figure 8.1.

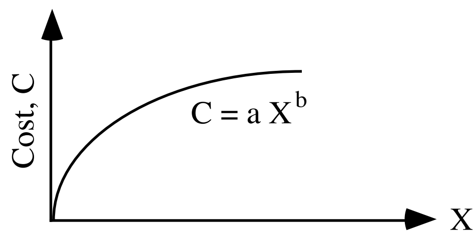

Figure 8.1: Typical Non-linear (Concave) Cost Function

In Figure 8.1, the constant exponent b determines the degree of non-linearity and is usually between 0.4 and 1.0. For b = 1, f(x) is linear and curvature increases as b decreases. As an example, $b \approx 0.6$ is a common value for pipelines.

Non-linearities may also occur in some types of constraints. For example, hydropower problems in which both flow rate (Q) and head on turbines (H) are variables require the non-linear constraint:

$$\text{Power generated} = f(Q \bullet H) \qquad \text{...[8.1]}$$

Various approaches exist for solving non-linear problems. One of these is to divide the nonlinear functions into several linear sections (piecewise linearization). The advantage of this approach is that we then have a linear problem to which any LP algorithm, such as LINGO, can be applied. Two approaches to this concept will be presented.

8.2 UNBOUNDED FUNCTION APPROACH

This method is limited to maximizing strictly concave functions, such as that illustrated in Figure 8.1, or minimizing convex functions such as that shown in Figure 8.2.

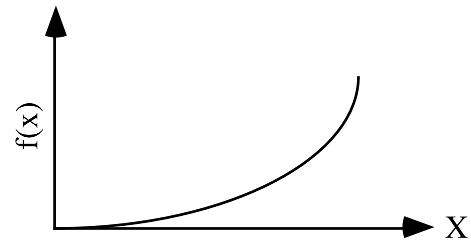

Figure 8.2: Convex Function

Assume that the problem is to maximize the concave function in Figure 8.3 subject to the constraint $X \leq 5$. The problem is, of course, trivial because the solution is $X = 5$. However, if there were 10 variables in both the objective and the constraint we could not draw a picture of the problem, but the concept which follows would still apply.

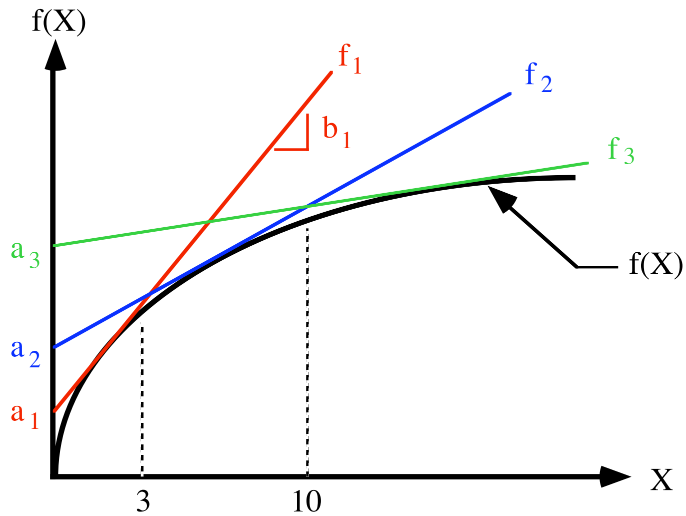

Figure 8.3: Unbounded Approach to Piecewise Linearization

Instead, write a new problem:

$$\text{Max } u \qquad \text{...[8.2]}$$

$$\text{s.t.:} \quad u \leq f_1 = a_1 + b_1 X \qquad \text{...[8.3]}$$

$$u \leq f_2 = a_2 + b_2 X \qquad \text{...[8.4]}$$

$$u \leq f_3 = a_3 + b_3 X \qquad \text{...[8.5]}$$

$$X \leq 5 \qquad \text{...[8.6]}$$

This is an LP problem because each new fi is linear and each $f_i \approx f(X)$ over some range of X. The LP solution will be $u = f_2(X)$ because it is less than f_1 or f_3 and, therefore, closer to f(X) when $3 \le X \le 10$. So the max value of $u = a_2 + b_2 (5)$. Note that in the range $0 \le X \le 3$, f_1 is the smallest and for $X \ge 10$, f_3 is smallest.

Similarly we could minimize a convex function:

$$\text{Min } f(X) \qquad \text{...[8.7]}$$

$$\text{s.t.:} \quad g(X) \ge b \qquad \text{...[8.8]}$$

by using

$$\text{Min } u \qquad \text{...[8.9]}$$

$$\text{s.t.:} \quad u \ge f_1 = a_1 + b_1 X \qquad \text{...[8.10]}$$

$$u \ge f_2 = a_2 + b_2 X \qquad \text{...[8.11]}$$

$$u \ge f_3 = a_3 + b_3 X \qquad \text{...[8.12]}$$

This is a very simple method that guarantees global optimum solutions, but is limited to the concave maximum or convex minimum restrictions given above. A more general approach (but one that guarantees only local optima without the same concave maximum/convex minimum restrictions) follows.

8.3 BOUNDED VARIABLE APPROACH

Consider the nonlinear function $f(X_1,X_2)$ which has been approximated by three nonlinear segments in the X_1 plane of Figure 8.4. The $f(X_2)$ portion is not shown but one can imagine similar linear segments in the X_2 direction which produce linear planes in three dimensions or linear hyperplanes in n dimensions.

The following notation demonstrates the method in n dimensions. The basic idea is to write the problem in terms of new artificial variables, W_{ji}, in which i identifies which original X_i is being divided into linear pieces. Variable W_{ji} is active between the end points j and j+1.

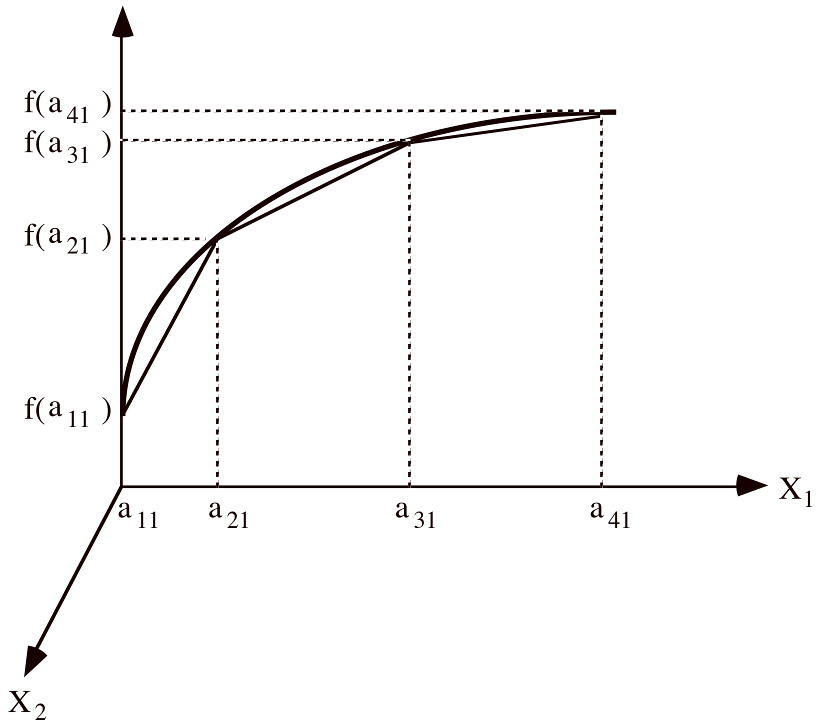

Figure 8.4: Bounded Variable Approach

Consider the original generalized problem:

$$\text{Max } Z = f(x_i) \qquad \text{...[8.13]}$$

$$\text{s.t.:} \quad g_k(x_i) \le b_k \qquad (k = 1, 2 \ldots L) \qquad \text{...[8.14]}$$

The piecewise linear problem can be written as follows:

$$\text{Max } Z = \sum_i \sum_j f(a_{ji})\, W_{ji} \qquad \text{...[8.15]}$$

$$\text{s.t.:} \quad \sum_j a_{ji}\, W_{ji} = X_i \qquad (i = 1,2, \ldots N) \qquad \text{...[8.16]}$$

$$\sum_j W_{ji} = 1 \qquad (i = 1,2, \ldots N) \qquad \text{...[8.17]}$$

$$\sum_i \sum_j g_k(a_{ji})\, W_{ji} \le b_k \qquad (k = 1,2, \ldots L) \qquad \text{...[8.18]}$$

Note that the last type of constraint is needed only for non-linear constraints; otherwise use the original $g_k(x_i) \le b_k$.

This method guarantees a global optimum solution only for maximization problems when the function to be maximized is concave, or for minimization problems when the function to be minimized is convex. However, it may be used for other functions if restricted basis entry (i.e., only two adjacent W_{ji} are allowed to enter the basis) software is available or if adjacent W_{ji} in are forced into the basis by iteratively using any LP software.

For example, if we were minimizing a concave function such as shown in Figure 8.5, the solution without restricted basis would be:

$$x^* = w_1 a_1 + w_4 a_4 \qquad \text{and } f(x) = f_1 \qquad \text{...[8.19]}$$

but this is incorrect because w_1 and w_4 are not adjacent and therefore $f(w_1,w_4)$ is not a good approximation of f(x). Restricted basis entry will prevent such solutions, and

$$x^* = w_2 a_2 + w_3 a_3 \qquad \text{and } f^* = f_2 \qquad \text{...[8.20]}$$

will therefore be selected for the constraint shown.

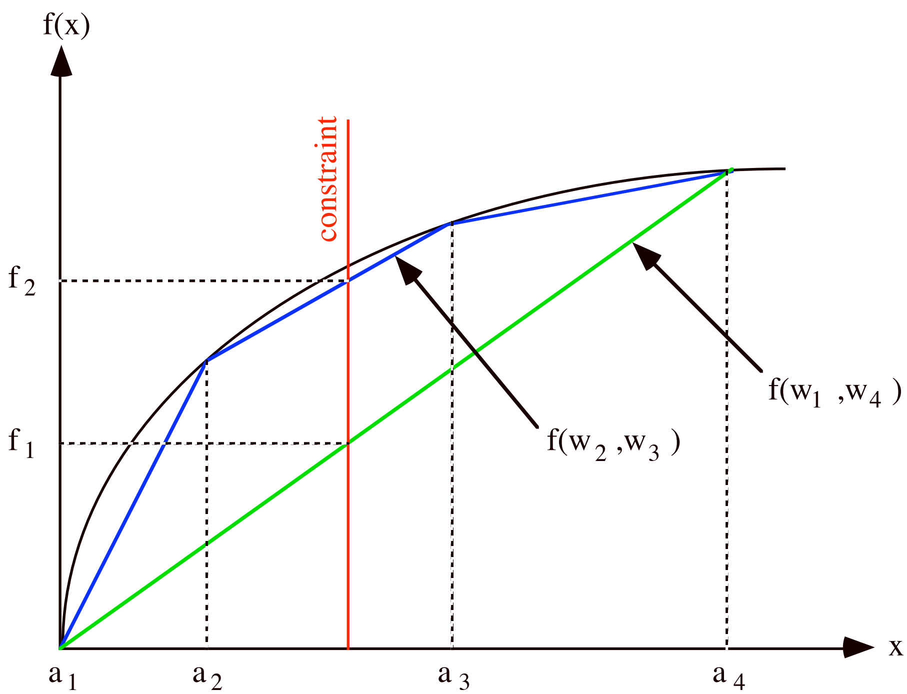

Figure 8.5: Restricted Basis Entry

8.4 EXAMPLE PROBLEMS

8.4.1 A Simple LP Problem

Develop an LP model for solution of the following non-linear problem by the bounded variable method.

$$\text{Max } Z = X_1 (5 - X_1^2) + X_2 (14 - 6 X_2) \qquad \text{...[8.21]}$$

s.t.: $X_1 + 4 X_2 \leq 18$ (Constraint 1) ...[8.22]

$6 X_1 + 2 X_2 \leq 26$ (Constraint 2) ...[8.23]

$X_2^2 \leq 5$ (Constraint 3) ...[8.24]

Solution:

Assume a grid interval of one unit and, for simplicity, an upper bound on X_1 and X_2 of 3. From this, a table of basic information about the problem can be constructed, as illustrated in Table 8.1.

Table 8.1: Calculation of Coefficients for Use in Piecewise Linearization

j	a_{ji}	$f(a_{j1})$	$f(a_{j2})$	$g_3(a_{j2})$
1	0	0	0	0
2	1	4	8	1
3	2	2	4	4
4	3	-12	-12	9

From the information tabulated in Table 8.1, the coefficients of an LP problem can be constructed, as illustrated in Table 8.2.

Table 8.2: Piecewise Linear LP Model Coefficients

Constraint	Variable										RHS	
	W_{11}	W_{21}	W_{31}	W_{41}	W_{12}	W_{22}	W_{32}	W_{42}	X_1	X_2		
Obj. Fn. Z =	0	4	2	-12	0	8	4	-12				
(1)									1	4	≤	18
(2)									6	2	≤	26
(3)					0	1	4	9			≤	5
	1	1	1	1							=	1
					1	1	1	1			=	1
	0	1	2	3					-1		=	0
					0	1	2	3		-1	=	0

8.4.2 Two-Reservoir Example Problem

Let us complete the discussion of piecewise linearization by considering the following multiple reservoir-multiple use design problem. This is a slightly modified version of a problem given by Dorfman in Maass et al. (1962).

Consider the sequential, two-reservoir problem shown in Figure 8.6. There are two uses for the water: irrigation and hydropower. Assume that optimization is to be based on average year flows of 3.3×10^6 acft during the wet season and 1.4×10^6 acft during the dry season, plus the wet and dry season inflows shown at two tributaries. The capacities of reservoirs A and B (Y_a and Y_b) are variables that are to be determined such that the total capacity will be filled during the wet season and completely released during the dry season. This policy will produce the river flows shown (or calculated) in various reaches. Irrigation volume (I) is a variable that must be allocated as shown during wet and dry seasons. The objective is to determine Y_a, Y_b, I, and E (where E is the energy generated per year in 10^9 KWH) such that net benefits from agriculture plus energy production are maximized. Flow through the turbine is related to energy as follows:

$$E = 0.144\ F \qquad \text{...[8.25]}$$

where F is flow in 10^6 acft. The demand for energy is uniform so that half must be generated during the wet season and half during the dry season. This means that:

$$F_w = [6.9 - Y_a - Y_b - 0.275\ I] \geq (0.5\ E / 0.144) = 3.47\ E \qquad \text{...[8.26]}$$

$$F_D = [3.9 + Y_a + Y_b - 0.125\ I] \geq (0.5\ E / 0.144) = 3.47\ E \qquad \text{...[8.27]}$$

Therefore, eliminating F_w and F_D, we have:

$$Y_a + Y_b + 0.275\ I + 3.47\ E \leq 6.9 \qquad \text{...[8.28]}$$

$$- Y_a - Y_b + 0.125\ I + 3.47\ E \leq 3.9 \qquad \text{...[8.29]}$$

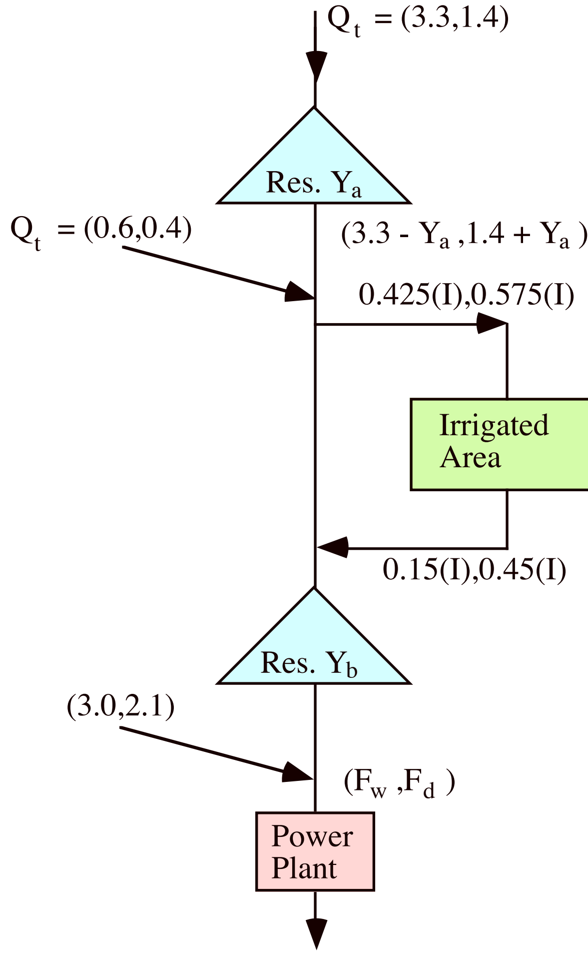

Figure 8.6: Two-Reservoir Irrigation and Power Problem

The objective is to maximize net benefits where:

$$NB = B_1(E) + B_2(I) - C_1(Y_a) - C_2(Y_b) - C_3(E) - C_4(I) \qquad ...[8.30]$$

These functions are as follows:

The benefit functions, B(E) and B(I), are the present value of yearly energy and agricultural production in $\$10^6$. The cost functions, C_i, are the capital costs of building the reservoirs, power plant, and irrigation systems in $\$10^6$. The cost functions are as follows:

$$C_1(Y_a) = \frac{43\ Y_a}{(1\ +\ 0.2\ Y_a)} \qquad ...[8.31]$$

$$C_2(Y_b) = \frac{47\ Y_b}{(1\ +\ 0.3\ Y_b)} \qquad ...[8.32]$$

$$C_3(E) = 30.6\ E - E^2 \qquad ...[8.33]$$

$$C_4(I) = 54\ I \qquad ...[8.34]$$

Note that the first three cost functions are non-linear due to economies of scale.

The benefit functions are:

$$B_1(E) = 250\ E \qquad \text{(marginal benefit is constant)} \qquad ...[8.35]$$

$$B_2(I) = 30(I) + 1045\ \ln[1 + 0.2(I)] \qquad ...[8.36]$$

The second benefit function has been derived by integrating a marginal benefit function.

Let us now proceed to set up the LP matrix for this problem using separable programming to linearize the objective function.

<u>Solution</u>:

The intuitive approach is to write the usual mass balance constraints at each reservoir plus each junction. We can do this, but in a simpler form since we assume both reservoirs will fill during the wet season and empty during the dry season. This implies S_t is not a variable ($S_1 = 0$, $S_2 = S_{max}$, $S_3 = 0$). Therefore, we can simply write constraints that prevent negative flows in each reach of the river. Analysis shows that several of these constraints are redundant so we use only the non-redundant set to reduce the size of the model.

Let:

Y_a, Y_b be the yields (S_{max}) of reservoirs A and B, respectively

I be the total annual irrigation demand (both seasons)

E be the annual production of electrical energy in units of 10^9 KWH

Table 8.3 summarizes the constraints that will be required.

Table 8.3: Summary of Constraints for the Two-Reservoir Problem Model

Constraint in Model	Constraints	River Reach
1 2	$3.3 - Y_a \geq \square$ $1.4 + Y_a \geq 0$ (this is redundant because $Y_a \geq 0$) The tributary adds flows, so both constraints are redundant.	1 2
3 4	$3.9\ Y_a - 0.425\ I \geq 0$ $1.8 + Y_a - 0.575\ I \geq \square$ Only return flows are added, so both constraints are redundant.	3 4
5 6	$3.9 - Y_a - Y_b - 0.275\ I \geq 0$ $1.8 + Y_a + Y_b - 0.125\ I \geq 0$ Only tributary flows are added, so these are redundant.	5 6
7, 8	Hydropower availability constraints, as previously given.	7

The objective function can be written as:

$$\text{Max } Z = f_1(Y_a) + f_2(Y_b) + f_3(I) + f4(E) \qquad \text{...[8.37]}$$

where

$$f_1(Y_a) = -\frac{43\ Y_a}{(1 + 0.2\ Y_a)} \qquad \text{...[8.38]}$$

$$f_2(Y_b) = -\frac{47\ Y_b}{(1 + 0.3\ Y_b)} \qquad \text{...[8.39]}$$

$$f_3(I) = -24\ I + 1045\ \ln(1 + 0.2\ I) \qquad \text{...[8.40]}$$

$$f_4(E) = 229.4\ E + E^2 \qquad \text{...[8.41]}$$

Table 8.4 presents the calculations for the piecewise linearization of the nonlinear functions.

Table 8.4: Calculation of Coefficients for Piecewise Linearization of the Two-Reservoir Hydropower and Irrigation Problem

j	(Y_a, Y_b, E, I) (a_{ji})	$f_1(Y_a)$ [or $f_1(a_{j1})$]	$f_2(Y_b)$ [or $f_2(a_{j2})$]	$f_3(I)$ [or $f_3(a_{j3})$]	$f_4(E)$ [or $f_4(a_{j4})$]
1	0	0	0	0	0
2	1	-35.8	-36.1	166.5	220.4
3	2	-61.0	-58.7	304	442.8
4	3	-80.6	-74.2	419	667.2
5	4	-95.5	-85.2	518	893.6
6	5		-94.0	604.3	1122
7	6			690	

It should be noted that in this example, we are maximizing an objective function that is not concave. Therefore, an LP solution will not be correct unless restricted basis entry is used. Most commercial LP algorithms have this capability (limiting basis variables to two adjacent W_{ji} for each i). However, if one is using an LP algorithm without this capability, restricted basis entry can be accomplished manually by solving several LP problems iteratively unless the problem is so large that the number of iterations becomes prohibitive. The procedure is to force one of the non-adjacent W_{ji} to 0, re-solve the problem, and continue until only adjacent W_{ji} are in the basis.

Bishop and Narayanan (1977) give a good example of the use of separable programming for a large non-linear planning problem.

8.5 SEPARABILITY

Note that the piecewise linear concepts presented work only because functions were 'separable' into terms which are functions of only a single non-linear variable. For example,

$$f(X_1,X_2) = 2\,X_1^2 + 3\,X_2^3 \qquad \text{...[8.42]}$$

is separable. However, a function such as

$$g(X_1,X_2) = X_1\,X_2 \qquad \text{...[8.43]}$$

(which is of the form of the Q•H hydropower function mentioned previously) presents difficulty because it cannot be linearized until X_1 and X_2 are separated. One way to do this is to do a transformation into two new variables, X_3 and X_4 as follows:

Let: $$X_3 = \frac{X_1 + X_2}{2} \qquad \text{...[8.44]}$$

$$X_4 = \frac{X_1 - X_2}{2} \qquad \text{...[8.45]}$$

Then

$$X_1 X_2 = X_3^2 - X_4^2 \qquad \text{...[8.46]}$$

because

$$\left(\frac{X_1 + X_2}{2}\right)^2 - \left(\frac{X_1 - X_2}{2}\right)^2 = \frac{X_1^2 + 2 X_1 X_2 + X_2^2}{4}$$

$$- \frac{X_1^2 - 2 X_1 X_2 + X_2^2}{4} = X_1 X_2 \qquad \text{...[8.47]}$$

Products of terms with other polynomials can also separated by using the generalized transformation:

$$X_1^a X_2^b = X_3^{2a} - X_4^{2b} \qquad \text{...[8.48]}$$

8.6 NON-LINEAR OPTIMIZATION SOFTWARE

The need for piecewise linearization is not as great now as it was in the past because software is available to optimize problems in non-linear form. This is true particularly for problems with a non-linear objective function, but with linear constraints. However, non-linear constraints still may need to be linearized. Also, no software exists to handle an integer programming problem with a nonlinear objective.

8.7 SIMULATION VERSUS OPTIMIZATION

The previous problem has only four real decision variables (until our linearization technique expanded it to 28 variables). This suggests that perhaps some simulation approach might be more appropriate since non-linearity would then present no difficulty. Two possible approaches are exhaustive enumeration and Monte Carlo simulation. The former will require a truly large number of calculations. If, for example, we wish to search over a grid with 0.1-unit increments, we need to have four nested "do-loops" where $0 \leq Y_a \leq 3.3$ requires 34 iterations, $0 \leq Y_b \leq 3.9$ requires (40)x(34) iterations, and E and I limits of 6 each require (40)x(34)x(61)x(61) = 5(106) calculations. One could alternatively search a coarser grid to locate the "good" solution vicinity and then do a finer grid search of this smaller area.

Consider, however, the Monte Carlo approach where a random number generator may be used to simultaneously vary the four decision variables. One could generate trial values of Y_a, Y_b, I, and E which are always within the possible range of each variable (but which may violate some of the six constraints). One could then test feasibility by checking the constraint set and save only those solutions which are both feasible and which improve the best previous objective function. Solutions of this type have been shown to be within 5.5 percent of the LP solution after 5000 iterations and within 2.8 percent of the LP solution after 20,000 iterations. Note that the LP solution also includes some error due to the difference between the piecewise linear functions of the true non-linear functions. Both LP and Monte Carlo methods seem to produce good models with reasonable computational effort. In a much larger problem, however, with hundreds--rather than four--decision variables, the Monte Carlo approach will become computationally prohibitive, while LP models can be readily solved with thousands of variables. Relatively recent advances in optimization techniques, called genetic algorithms (that are similar to Monte Carlo methods in that they employ random but controlled search methods) show considerable promise for use in water resources planning and management. These are discussed in a later chapter.

8.8 PROBLEMS

1. Solve Problem 1 from Chapter 7 as a piecewise linear problem. The problem is the same as discussed in Chapter 7, except that the capital cost function is continuous and non-linear as shown in Table 8.5, below. The reservoir may be any capacity between 0 and 900.

Table 8.5: Reservoir Capacity and Capital Cost Data

Reservoir Capacity (acft)	Capital Cost ($/year)
0	0
350	5,000
700	15,000
900	18,000

2. Develop a piecewise linear model of the multiple use-two reservoir problem presented in this chapter. Then solve the model using LINGO. Note that manual iterations to achieve restricted basis entry will be necessary since you are maximizing functions that are not all concave. One can avoid the necessity of these manual operations with judicious use of integer variables to control restricted basis entry.

   Solve the same problem using the non-linear programming capability of LINGO instead of piecewise linearization.

3. Solve the same reservoir problem by using a Monte Carlo simulation approach. Compare the answers for an increasing number of trials. Also compare the answers of Exercises 2 and 3. Which is more correct? Is there any error in the Exercise 2 answer?