# CVEN 5161 

# Advanced Mechanics of Materials I 

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Revised Version of Class Notes

Fall 2003

## Chapter 1

## Preliminaries

The mathematical tools behind stress and strain are housed in Linear Algebra and Vector and Tensor Analysis in particular. For this reason let us revisit established concepts which hopefully provide additional insight into matrix analysis beyond the mere mechanics of elementary matrix manipulation.

### 1.1 Vector and Tensor Analysis

## (a) Cartesian Description of Vectors:

Forces, displacements, velocities and accelerations are objects $\boldsymbol{F}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a} \in \Re^{3}$ which may be represented in terms of a set of base vectors and their components. In the following we resort to cartesian coordinates in which the base vectors $\boldsymbol{e}_{i}$ form an orthonormal set which satisfies

$$
\boldsymbol{e}_{i} \cdot \boldsymbol{e}_{j}=\delta_{i j} \quad \forall i, j=1,2,3 \quad \text { where } \quad\left[\delta_{i j}\right]=\mathbf{1}=\left[\begin{array}{lll}
1 & 0 & 0  \tag{1.1}\\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

where the Kronecker symbol $\left[\delta_{i j}\right]=\mathbf{1}$ designates the unit tensor. Consequently, the force vector may be expanded in terms of its components and base vectors,

$$
\begin{equation*}
\boldsymbol{F}=F_{1} \boldsymbol{e}_{1}+F_{2} \boldsymbol{e}_{2}+F_{3} \boldsymbol{e}_{3}=\sum_{i}^{3} F_{i} \boldsymbol{e}_{i} \rightarrow F_{i} \boldsymbol{e}_{i} \tag{1.2}
\end{equation*}
$$

The last expression is a short hand index notation in which repeated indices infer summation over 1,2,3.

In matrix notation, the inner product may be written in the form,

$$
\boldsymbol{F}=\left[F_{1}, F_{2}, F_{3}\right]^{t}\left[\begin{array}{l}
\boldsymbol{e}_{1}  \tag{1.3}\\
\boldsymbol{e}_{2} \\
\boldsymbol{e}_{3}
\end{array}\right] \rightarrow\left[F_{i}\right]=\left[\begin{array}{c}
F_{1} \\
F_{2} \\
F_{3}
\end{array}\right]
$$

where we normally omit the cartesian base vectors and simply represent a vector in terms of its components (coordinates).
(b) Scalar Product (Dot Product) of Two Vectors: $(\boldsymbol{F} \cdot \boldsymbol{u}) \in \Re^{3}$

The dot operation generates the scalar:

$$
\begin{equation*}
W=(\boldsymbol{F} \cdot \boldsymbol{u})=\left(F_{i} \boldsymbol{e}_{i}\right) \cdot\left(u_{j} \boldsymbol{e}_{j}\right)=F_{i} u_{j} \delta_{i j}=F_{i} u_{i} \tag{1.4}
\end{equation*}
$$

Using matrix notation, the inner product reads $\boldsymbol{F}^{t} \boldsymbol{u}=F_{i} u_{i}$, where $\boldsymbol{F}^{t}$ stands for a row vector and $\boldsymbol{u}$ for a column vector. The mechanical interpretation of the scalar product is a work or energy measure with the unit $[1 \mathrm{~J}=1 \mathrm{Nm}]$. The magnitude of the dot product is evaluated according to,

$$
\begin{equation*}
W=|\boldsymbol{F}||\boldsymbol{u}| \cos \theta \tag{1.5}
\end{equation*}
$$

where the absolute values $|\boldsymbol{F}|=(\boldsymbol{F} \cdot \boldsymbol{F})^{\frac{1}{2}}$ and $|\boldsymbol{u}|=(\boldsymbol{u} \cdot \boldsymbol{u})^{\frac{1}{2}}$ are the Euclidean lengths of the two vectors. The angle between two vectors is defined as

$$
\begin{equation*}
\cos \theta=\frac{(\boldsymbol{F} \cdot \boldsymbol{u})}{|\boldsymbol{F}||\boldsymbol{u}|} \tag{1.6}
\end{equation*}
$$

The two vectors are orthogonal when $\theta= \pm \frac{\pi}{2}$,

$$
\begin{equation*}
(\boldsymbol{F} \cdot \boldsymbol{u})=0 \quad \text { if } \quad \boldsymbol{F} \neq \mathbf{0} \quad \text { and } \quad \boldsymbol{u} \neq \mathbf{0} \tag{1.7}
\end{equation*}
$$

In other terms there is no work done when the force vector is orthogonal to the displacement vector.

Note: Actually, the mechanical work is the line integral of the scalar product between the force vector times the displacement rate, $W=\int \boldsymbol{F} \cdot d \boldsymbol{u}$. Hence the previous expressions were based on the assumption that the change of work along the integration path is constant.
(c) Vector Product (Cross Product) of Two Vectors: $(\boldsymbol{x} \times \boldsymbol{y}) \in \Re^{3}$ :

The cross product generates the vector $\boldsymbol{z}$ which is orthogonal to the plane spanned by the two vectors $\boldsymbol{x}, \boldsymbol{y}$ according to the right hand rule:

$$
\begin{equation*}
\boldsymbol{z}=\boldsymbol{x} \times \boldsymbol{y}=x_{i} y_{j}\left(\boldsymbol{e}_{i} \times \boldsymbol{e}_{j}\right) \quad \text { where } \quad|\boldsymbol{z}|=|\boldsymbol{x}||\boldsymbol{y}| \sin \theta \tag{1.8}
\end{equation*}
$$

In 3-d the determinant scheme is normally used for the numerical evaluation of the cross product. Its components may be written in index notation with respect to cartesian coordinates

$$
\begin{equation*}
z_{p}=\epsilon_{p q r} x_{q} y_{r} \tag{1.9}
\end{equation*}
$$

The permutation symbol $\epsilon_{p q r}=0,1,-1$ is zero when any two indices are equal, it is one when $p, q$, r are even permutations of $1,2,3$, and it is minus one when they are odd.
The geometric interpretation of the cross product is the area spanned by the two vectors, $|\boldsymbol{z}|=A$. In other terms the determinant is a measure of the area subtended by the two vectors $\boldsymbol{x} \times \boldsymbol{y}$, where

$$
\boldsymbol{z}=\operatorname{det}\left[\begin{array}{lll}
\boldsymbol{e}_{1} & \boldsymbol{e}_{2} & \boldsymbol{e}_{3}  \tag{1.10}\\
x_{1} & x_{2} & x_{3} \\
y_{1} & y_{2} & y_{3}
\end{array}\right]
$$

The scalar triple product $V=\boldsymbol{z} \cdot(\boldsymbol{x} \times \boldsymbol{y})$ measures the volume spanned by the three vectors:

$$
V=|\boldsymbol{z}| \cos \psi|\boldsymbol{x}||\boldsymbol{y}| \sin \theta=\operatorname{det}\left[\begin{array}{lll}
x_{1} & x_{2} & x_{3}  \tag{1.11}\\
y_{1} & y_{2} & y_{3} \\
z_{1} & z_{2} & z_{3}
\end{array}\right]
$$

(d) Tensor Product (Dyadic Product) of Two Vectors: $\boldsymbol{u} \otimes \boldsymbol{v} \in \Re^{3}$ :

The tensor product of two vectors generates a second order tensor (i.e. a matrix of rank-one) $\boldsymbol{A}^{(1)}$ :

$$
\begin{equation*}
\boldsymbol{A}^{(1)}=\boldsymbol{u} \otimes \boldsymbol{v}=u_{i} \boldsymbol{e}_{i} \otimes v_{j} \boldsymbol{e}_{j}=u_{i} v_{j} \boldsymbol{e}_{i} \otimes \boldsymbol{e}_{j}=\left[u_{i} v_{j}\right] \tag{1.12}
\end{equation*}
$$

In matrix notation the tensor product writes as $\boldsymbol{u} \otimes \boldsymbol{v}=\left[u_{i} v_{j}\right]=\boldsymbol{u} \boldsymbol{v}^{t}$. Expanding, we have

$$
\boldsymbol{u} \otimes \boldsymbol{v}=\left[u_{i} v_{j}\right]=\left[\begin{array}{lll}
u_{1} v_{1} & u_{1} v_{2} & u_{1} v_{3}  \tag{1.13}\\
u_{2} v_{1} & u_{2} v_{2} & u_{2} v_{3} \\
u_{3} v_{1} & u_{3} v_{2} & u_{3} v_{3}
\end{array}\right]
$$

The linear combination of three dyads is called a 'dyadic' which may be used to generate a second order tensor of full rank three, $\boldsymbol{A}^{(3)}$ :

$$
\begin{equation*}
\boldsymbol{A}^{(3)}=\sum_{k=1}^{3} a_{k} \boldsymbol{u}_{k} \otimes \boldsymbol{v}_{k} \tag{1.14}
\end{equation*}
$$

Spectral representation of a second order tensor $\boldsymbol{A}^{(3)}$ expresses the tensor in terms of eigenvalues $\lambda_{i}$ and eigenvectors. They form an orthogonal frame which may be normalized by $\boldsymbol{e}_{i} \cdot \boldsymbol{e}_{j}=\mathbf{1}$ such that,

$$
\begin{equation*}
\boldsymbol{A}^{(3)}=\sum_{i=1}^{3} \lambda_{i} \boldsymbol{e}_{i} \otimes \boldsymbol{e}_{i} \tag{1.15}
\end{equation*}
$$

Note: the tensor product increases the order of of the resulting tensor by a factor two. Hence the tensor product of two vectors (two tensors of order one) generates a second order tensor, and the tensor product of two second order tensors generates a fourth order tensor, etc.

## (e) Coordinate Transformations:

Transformation of the components of a vector from a proper orthonormal coordinate system $\boldsymbol{e}_{1}, \boldsymbol{e}_{2}, \boldsymbol{e}_{3}$ into another proper orthonormal system $\boldsymbol{E}_{1}, \boldsymbol{E}_{2}, \boldsymbol{E}_{3}$ involves the operator $\boldsymbol{Q}$ of direction cosines. In 3 -d we have,

$$
\begin{equation*}
\boldsymbol{Q}=\cos \left(\boldsymbol{E}_{i} \cdot \boldsymbol{e}_{j}\right) \tag{1.16}
\end{equation*}
$$

where the matrix of direction cosines is comprised of,

$$
\begin{array}{c|ccc} 
& \boldsymbol{e}_{1} & \boldsymbol{e}_{2} & \boldsymbol{e}_{3} \\
\hline \boldsymbol{E}_{1} & \cos \left(E_{1} e_{1}\right) & \cos \left(E_{1} e_{2}\right) & \cos \left(E_{1} e_{3}\right)  \tag{1.17}\\
\boldsymbol{E}_{2} & \cos \left(E_{2} e_{1}\right) & \cos \left(E_{2} e_{2}\right) & \cos \left(E_{2} e_{3}\right) \\
\boldsymbol{E}_{3} & \cos \left(E_{3} e_{1}\right) & \cos \left(E_{3} e_{2}\right) & \cos \left(E_{3} e_{3}\right)
\end{array}
$$

The $\boldsymbol{Q}$-operator forms a proper orthonormal transformation, $\boldsymbol{Q}^{-1}=\boldsymbol{Q}^{t}$ i.e. it satisfies the conditions,

$$
\begin{equation*}
\boldsymbol{Q}^{t} \cdot \boldsymbol{Q}=\mathbf{1} \quad \text { and } \quad \operatorname{det} \boldsymbol{Q}=1 \tag{1.18}
\end{equation*}
$$

In the case of $2-\mathrm{d}$, this transformation results in a plane rotation around the $x_{3}$-axis, i.e.

$$
\boldsymbol{Q}=\left[\begin{array}{ccc}
\cos \theta & \sin \theta & 0  \tag{1.19}\\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{array}\right]
$$

where $\theta \geq 0$ for counterclockwise rotations according the right hand rule. Consequently, the transformation of the components of the vector $\boldsymbol{F}$ from one coordinate system into another,

$$
\begin{equation*}
\tilde{\boldsymbol{F}}=\boldsymbol{Q} \cdot \boldsymbol{F} \quad \text { and inversely } \quad \boldsymbol{F}=\boldsymbol{Q}^{t} \cdot \tilde{\boldsymbol{F}} \tag{1.20}
\end{equation*}
$$

The $(3 \times 3)$ dyadic forms an array $\boldsymbol{A}$ of nine scalars which follows the transformation rule of second order tensors,

$$
\begin{equation*}
\tilde{\mathbf{A}}=\mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{Q}^{t} \quad \text { and inversely } \quad \mathbf{A}=\mathbf{Q}^{t} \cdot \tilde{\mathbf{A}} \cdot \mathbf{Q} \tag{1.21}
\end{equation*}
$$

This distinguishes the array $\boldsymbol{A}$ to form a second order tensor.

## (f) The Stress Tensor:

Considering the second order stress tensor as an example when $\boldsymbol{A}=\boldsymbol{\sigma}$ then the linear transformation results in the Cauchy theorem, that mapping of the stress tensor onto the plane with the normal $\boldsymbol{n}$ results in the traction vector $\boldsymbol{t}$ :

$$
\begin{equation*}
\boldsymbol{\sigma}^{t} \cdot \boldsymbol{n}=\boldsymbol{t} \quad \text { or } \quad \sigma_{j i} n_{j}=t_{i} \tag{1.22}
\end{equation*}
$$

This transformation may be viewed as a projection of the stress tensor onto the plane with the normal vector $\boldsymbol{n}^{t}=\left[n_{1}, n_{2}, n_{3}\right]$, where

$$
\boldsymbol{\sigma}=\left[\begin{array}{lll}
\sigma_{11} & \sigma_{12} & \sigma_{13}  \tag{1.23}\\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{array}\right]
$$

with the understanding that $\boldsymbol{\sigma}=\boldsymbol{\sigma}^{t}$ is symmetric in the case of Boltzmann continua. The Cauchy theorem states that the state of stress is defined uniquely in terms of the traction vectors on three orthogonal planes $\boldsymbol{t}_{1}, \boldsymbol{t}_{2}, \boldsymbol{t}_{3}$ which form the nine entries in the stress tensor. Given the stress tensor, the traction vector is uniquely defined on any arbitrary plane with the normal $\boldsymbol{n}$, the components of which are,

$$
\boldsymbol{t}=\boldsymbol{\sigma}^{t} \cdot \boldsymbol{n}=\left[\begin{array}{c}
\sigma_{11} n_{1}+\sigma_{21} n_{2}+\sigma_{31} n_{3}  \tag{1.24}\\
\sigma_{12} n_{1}+\sigma_{22} n_{2}+\sigma_{32} n_{3} \\
\sigma_{13} n_{1}+\sigma_{23} n_{2}+\sigma_{33} n_{3}
\end{array}\right]
$$

The traction vector $\boldsymbol{t}$ may be decomposed into normal and tangential components on the plane $\boldsymbol{n}$,

$$
\begin{equation*}
\sigma_{n}=\boldsymbol{n} \cdot \boldsymbol{t}=\boldsymbol{n} \cdot \boldsymbol{\sigma}^{t} \cdot \boldsymbol{n} \tag{1.25}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\tau_{n}\right)^{2}=|\boldsymbol{t}|^{2}-\sigma_{n}^{2}=\boldsymbol{t} \cdot \boldsymbol{t}-(\boldsymbol{n} \cdot \boldsymbol{t})^{2} \tag{1.26}
\end{equation*}
$$

which are the normal stress and the tangential stress components on that plane.

## (g) Eigenvalues and Eigenvectors:

There exist a nonzero vector $\boldsymbol{n}_{p}$ such that the linear transformation $\boldsymbol{\sigma}^{t} \cdot \boldsymbol{n}_{p}$ is a multiple of $\boldsymbol{n}_{p}$. In this case $\boldsymbol{n} \| \boldsymbol{t}$ we talk about principal direction $\boldsymbol{n}=\boldsymbol{n}_{p}$ in which the tangential shear components vanish,

$$
\begin{equation*}
\boldsymbol{\sigma}^{t} \cdot \boldsymbol{n}_{p}=\lambda \boldsymbol{n}_{p} \tag{1.27}
\end{equation*}
$$

The eigenvalue problem is equivalent to stating that

$$
\begin{equation*}
\left[\boldsymbol{\sigma}^{t}-\lambda \mathbf{1}\right] \cdot \boldsymbol{n}_{p}=\mathbf{0} \tag{1.28}
\end{equation*}
$$

For a non-trivial solution $\boldsymbol{n}_{p} \neq \mathbf{0}$ the matrix $\left(\boldsymbol{\sigma}^{t}-\lambda \mathbf{1}\right)$ must be singular. Consequently, $\operatorname{det}\left(\boldsymbol{\sigma}^{t}-\lambda \mathbf{1}\right)=0$ generates the characteristic polynomial

$$
\begin{equation*}
p(\lambda)=\operatorname{det}\left(\boldsymbol{\sigma}^{t}-\lambda \mathbf{1}\right)=\lambda^{3}-I_{1} \lambda^{2}-I_{2} \lambda-I_{3}=0 \tag{1.29}
\end{equation*}
$$

where the three principal invariants are,

$$
\begin{gather*}
I_{1}=\operatorname{tr} \boldsymbol{\sigma}=\sigma_{i i}=\sigma_{1}+\sigma_{2}+\sigma_{3}  \tag{1.30}\\
I_{2}=\frac{1}{2}\left[\operatorname{tr} \boldsymbol{\sigma}^{2}-(\operatorname{tr} \boldsymbol{\sigma})^{2}\right]=\frac{1}{2}\left[\sigma_{i j} \sigma_{i j}-\sigma_{i i}^{2}\right]=-\left[\sigma_{1} \sigma_{2}+\sigma_{2} \sigma_{3}+\sigma_{3} \sigma_{1}\right]  \tag{1.31}\\
I_{3}=\operatorname{det} \boldsymbol{\sigma}=\sigma_{1} \sigma_{2} \sigma_{3} \tag{1.32}
\end{gather*}
$$

According to the fundamental theorem of algebra, a polynomial of degree 3 has exactly 3 roots, thus each matrix $\boldsymbol{\sigma} \in \Re^{3}$ has 3 eigenvalues $\lambda_{1}=\sigma_{1}, \lambda_{2}=\sigma_{2}, \lambda_{3}=\sigma_{3}$. If a polynomial with real coefficients has some non-real complex zeroes, they must occur in conjugate pairs.
Note: all three eigenvalues are real when the stress tensor $\boldsymbol{\sigma}=\boldsymbol{\sigma}^{t}$ is symmetric.
Further, $\operatorname{det}\left(\boldsymbol{\sigma}^{t}-\lambda \mathbf{1}\right)=(-1)^{3} \operatorname{det}\left(\lambda \mathbf{1}-\boldsymbol{\sigma}^{t}\right)$, thus the roots of both characteristic equations are the same.

## (i) Similarity Equivalence:

Similarity transformations, i.e. triple products of the form

$$
\begin{equation*}
\tilde{\boldsymbol{\sigma}}=\boldsymbol{S}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{S} \tag{1.33}
\end{equation*}
$$

preserve the spectral properties of $\boldsymbol{\sigma}$. In other terms, if $\tilde{\boldsymbol{\sigma}} \sim \boldsymbol{\sigma}$, then the characteristic polynomial of $\tilde{\boldsymbol{\sigma}}$ is the same as that of $\boldsymbol{\sigma}$.

$$
\begin{equation*}
p(\lambda)=\operatorname{det}(\tilde{\boldsymbol{\sigma}}-\lambda \mathbf{1})=\operatorname{det}\left(\boldsymbol{S}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{S}-\lambda \boldsymbol{S}^{-1} \cdot \boldsymbol{S}\right)=\operatorname{det}(\boldsymbol{\sigma}-\lambda \mathbf{1}) \tag{1.34}
\end{equation*}
$$

## (j) Orthonormal or Unitary Equivalence:

A matrix $\boldsymbol{U} \in \Re^{3}$ is unitary (real orthogonal) if

$$
\begin{equation*}
\boldsymbol{U}^{t} \cdot \boldsymbol{U}=\mathbf{1} \quad \text { and } \quad \boldsymbol{U}^{t}=\boldsymbol{U}^{-1} \tag{1.35}
\end{equation*}
$$

with $\operatorname{det} \boldsymbol{U}=1$ for proper orthonormal transformations. Consequently each unitary transformation is also a similarity transformation, $\tilde{\boldsymbol{\sigma}} \sim \boldsymbol{\sigma}$, where

$$
\begin{equation*}
\tilde{\boldsymbol{\sigma}}=\boldsymbol{U}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{U}=\boldsymbol{U}^{t} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{U} \tag{1.36}
\end{equation*}
$$

but not vice versa.
Note: Unitary equivalence distinguishes second order tensors from matrices of order three, since it preserves the length of the tensor $\boldsymbol{\sigma}$ in any coordinate system. In other terms, a zero length tensor in one coordinate system will have zero length in any other coordinate system.

## (k) Cayley-Hamilton Theorem:

This theorem states that every square matrix satisfies its own characteristic equation. In other terms the scalar polynomial $p(\lambda)=\operatorname{det}\left(\lambda \mathbf{1}-\boldsymbol{\sigma}^{t}\right)$ also holds for the matrix polynomial $p(\boldsymbol{\sigma})$. One important application of the Cayley-Hamilton theorem is to express powers of the matrix $\boldsymbol{\sigma}^{k}$ as linear combination of $\mathbf{1}, \boldsymbol{\sigma}, \boldsymbol{\sigma}^{2}$ for $k \geq 3$. In short, the tensor satisfies its characteristic equation.

$$
\begin{equation*}
p(\lambda)=\boldsymbol{\sigma}^{3}-I_{1} \boldsymbol{\sigma}^{2}-I_{2} \boldsymbol{\sigma}-I_{3}=0 \tag{1.37}
\end{equation*}
$$

## (l) Examples of Unitary (Proper Orthonormal) Transformations:

1. Plane Rotation:
$\boldsymbol{U}(\theta, i, j)$ is the identity matrix where the $i i, j j$ entries are replaced by $\cos \theta$ and the $i j$ entry by $-\sin \theta$ and $j i$ by $\sin \theta$.

$$
\boldsymbol{U}(\theta, i, j)=\left[\begin{array}{ccc}
\cos \theta & 0 & -\sin \theta  \tag{1.38}\\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{array}\right]
$$

It is apparent that this corresponds to the plane coordinate rotation discussed earlier on, where $\boldsymbol{Q}=\boldsymbol{U}^{t}$.
Note: Successive application of plane rotations reduces $\boldsymbol{\sigma}$ to diagonal form (Jacobi and Givens method to extract eigenvalues).
2. Householder Transformation:
$\boldsymbol{U}_{w}=\mathbf{1}-t \boldsymbol{w} \boldsymbol{w}^{t}$ is a rank one update of the unit matrix in the form of a reflection such that

$$
\begin{equation*}
\boldsymbol{U}_{w}=\mathbf{1}-2 \boldsymbol{w} \boldsymbol{w}^{t} \quad \text { with } \quad t=2\left(\boldsymbol{w}^{t} \boldsymbol{w}\right)^{-1} \quad \text { and } \quad \boldsymbol{w}^{t} \boldsymbol{w}=1 \tag{1.39}
\end{equation*}
$$

Consequently, the Householder transformation acts as identity transformation, $\boldsymbol{U}_{w} \boldsymbol{x}=\boldsymbol{x}$ if $\boldsymbol{x} \perp \boldsymbol{w}$ and $\boldsymbol{U}_{w} \boldsymbol{w}=-\boldsymbol{w}$ is a reflection.
Note: Successive application of Householder transformations reduces $\boldsymbol{\sigma}$ to Hessenberg form (in case of symmetry to a tri-diagonal form).

## Chapter 2

## Differential Equilibrium:

The balance laws of continuum mechanics comprise statements as follows:

1. Balance of Linear Momentum.
2. Balance of Angular Momentum.
3. Balance of Mass.
4. Balance of Energy (First Law of Thermodynamics).
" Forces" are measured indirectly through their action on deformable solids. Distributed forces include:

- b : body force/unit volume(i. e. density).
- $\mathbf{t}_{n}$ : surface traction.

Integrating the distributed forces over part I of the body $\mathcal{B}$ defines the resultant force vector

$$
\begin{equation*}
\mathbf{f}_{I}=\int_{\mathcal{B}_{\mathcal{I}}} \mathbf{b} d v+\int_{\partial \mathcal{B}_{\mathcal{I}}} \mathbf{t}_{n} d a \tag{2.1}
\end{equation*}
$$

The Cauchy Lemma : states pointwise balance of surface tractions across any surface in the interior of the body.

$$
\begin{equation*}
\mathbf{t}_{n}(\mathbf{x}, \mathbf{n})+\mathbf{t}_{n}(\mathbf{x},-\mathbf{n})=\mathbf{0} \tag{2.2}
\end{equation*}
$$

or

$$
\begin{equation*}
\mathbf{t}_{n}(\mathbf{x}, \mathbf{n})=-\mathbf{t}_{n}(\mathbf{x},-\mathbf{n}) \tag{2.3}
\end{equation*}
$$

### 2.1 The Cauchy Theorem:

The stress tensor is a linear mapping of the stress vector $\mathbf{t}_{n}$ onto the normal vector $\mathbf{n}$.

$$
\begin{equation*}
\mathbf{t}_{n}=\boldsymbol{\sigma}^{t}(\mathbf{x}) \cdot \mathbf{n} \tag{2.4}
\end{equation*}
$$

In indicial notation,

$$
\begin{equation*}
t_{i}=\sigma_{j i} n_{j} \tag{2.5}
\end{equation*}
$$

Considering elementary tetrahedron:

$$
\begin{equation*}
\mathbf{t}_{n}=\mathbf{n}_{1} \mathbf{t}_{1}+\mathbf{n}_{2} \mathbf{t}_{2}+\mathbf{n}_{3} \mathbf{t}_{3} \tag{2.6}
\end{equation*}
$$

The stress tensor $\boldsymbol{\sigma}$ is simply composed of the coordinates of stress vectors on three mutually orthogonal planes

$$
\boldsymbol{\sigma}=\left[\begin{array}{lll}
\sigma_{11} & \sigma_{12} & \sigma_{13}  \tag{2.7}\\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{array}\right]
$$

In our notation of $\sigma_{i j}$, the first subscript $i$ refers to normal direction of the area element and the subscript $j$ refers to the direction of the traction.

Considering the equilibrium in $x_{1}$ direction : $\sum f_{x 1}=0$.

$$
\begin{equation*}
\mathbf{t}_{1} d a=\sigma_{11} n_{1} d a+\sigma_{21} n_{2} d a+\sigma_{31} n_{3} d a \tag{2.8}
\end{equation*}
$$

then

$$
\begin{equation*}
t_{1}=\sigma_{11} n_{1}+\sigma_{21} n_{2}+\sigma_{31} n_{3} \tag{2.9}
\end{equation*}
$$

With the help of the Divergence Theorem we find

$$
\begin{equation*}
\int_{\partial \mathcal{B}} \sigma_{j i} n_{j} d a=\int_{\mathcal{B}} \sigma_{j i, j} d v \tag{2.10}
\end{equation*}
$$

The equation of motion by Cauchy states the balance between the body forces and surface tractions when inertia forces remain negligible:

$$
\begin{equation*}
\int_{\mathcal{B}} b_{i} d v+\int_{\partial \mathcal{B}} t_{i}^{n} d a=0 \tag{2.11}
\end{equation*}
$$

can be rewritten as

$$
\begin{equation*}
\sigma_{j i, j}+b_{i}=0 \tag{2.12}
\end{equation*}
$$

"Cauchy's First Theorem" states pointwise equilibrium in the interior of the body.

In the dynamic case, the balance equations generalize to

$$
\begin{equation*}
\sigma_{j i, j}+b_{i}=\frac{D}{D t}\left(\rho \dot{x_{i}}\right) \tag{2.13}
\end{equation*}
$$

### 2.2 Balance of Linear Momentum:

Linear momentum is defined as:

$$
\begin{equation*}
\mathbf{i}=\int_{\mathcal{B}_{\mathcal{I}}} \rho \dot{\mathbf{u}} d v \tag{2.14}
\end{equation*}
$$

Application of Newton's second law $\sum \mathbf{f}=\mathbf{m} \cdot \mathbf{a}$ to the control volume of the body

$$
\begin{equation*}
\frac{D}{D t} \mathbf{i}=\mathbf{f} \tag{2.15}
\end{equation*}
$$

"Dynamic equilibrium" or the balance of linear momentum may be expressed as

$$
\begin{equation*}
\int_{\mathcal{B}_{I}} \frac{D}{D t}(\rho \dot{\mathbf{u}}) d v=\int_{\partial \mathcal{B}_{\mathcal{I}}} \mathbf{b} d v+\int_{\mathcal{B}_{I}} \mathbf{t}_{n} d a \tag{2.16}
\end{equation*}
$$

or in terms of

$$
\begin{equation*}
\int_{\mathcal{B}_{I}}\left(\mathbf{b}-\frac{D}{D t}(\rho \dot{\mathbf{x}})\right) d v+\int_{\partial \mathcal{B}} \mathbf{t}_{n} d a=0 \tag{2.17}
\end{equation*}
$$

### 2.3 Balance of Angular Momentum

Angular momentum involves

$$
\begin{equation*}
\mathbf{h}_{0}=\int_{\mathcal{B}_{\mathcal{I}}}(\mathbf{x} \times \rho \dot{\mathbf{u}}) d v \tag{2.18}
\end{equation*}
$$

where the pole is assumed to coincide with the origin $\mathbf{x}_{0}=0$. The moment of the distributed forces is

$$
\begin{equation*}
\mathbf{m}_{0}=\int_{\mathcal{B}}(\mathbf{x} \times \mathbf{b}) d v+\int_{\partial \mathcal{B}}\left(\mathbf{x} \times \mathbf{t}_{n}\right) d a \tag{2.19}
\end{equation*}
$$

The balance of angular momentum states

$$
\begin{array}{r}
\frac{D \mathbf{h}_{0}}{D t}=\mathbf{m}_{0} \\
\frac{D}{D t} \int_{\mathcal{B}}(\mathbf{x} \times \rho \dot{\mathbf{u}}) d v=\int_{\mathcal{B}}(\mathbf{x} \times \mathbf{b}) d v+\int_{\partial \mathcal{B}}\left(\mathbf{x} \times \mathbf{t}_{n}\right) d a \tag{2.21}
\end{array}
$$

The divergence theorem yields for the last term above

$$
\begin{equation*}
\int_{\mathcal{B}}\left[\mathbf{x} \times\left(\mathbf{b}+\operatorname{div} \boldsymbol{\sigma}^{t}-\rho \ddot{\mathbf{u}}\right)\right] d v+2 \int_{\mathcal{B}}\left(\mathbf{1} \times \boldsymbol{\sigma}^{t}\right) d v=0 \tag{2.22}
\end{equation*}
$$

Application of the first theorem of Cauchy : $\mathbf{b}+\operatorname{div} \boldsymbol{\sigma}^{t}-\rho \frac{D}{D t} \dot{\mathbf{u}}=0$, we get

$$
\begin{equation*}
1 \times \sigma^{t}=0 \rightarrow \sigma=\sigma^{t} \tag{2.23}
\end{equation*}
$$

which states that the stress tensors symmetric, i.e. $\boldsymbol{\sigma}^{t}=\boldsymbol{\sigma}$, i.e. the "Boltzmann Postulate" of a symmetric stress tensor.
In index notation :

$$
\begin{equation*}
e_{i j k} \sigma_{j k}=0 \rightarrow \sigma_{j k}=\sigma_{k j} \tag{2.24}
\end{equation*}
$$

this results in

$$
\begin{equation*}
\sigma_{23}-\sigma_{32}=0 ; \sigma_{31}-\sigma_{13}=0 ; \sigma_{12}-\sigma_{21}=0 \tag{2.25}
\end{equation*}
$$

### 2.4 Alternative Stress Measures:

Consider that the force vector is the same on the deformed and undeformed surface areas

$$
\begin{equation*}
\mathbf{f}=\mathbf{t}_{n} d a=\boldsymbol{\sigma}^{t} \mathbf{n} d a=\boldsymbol{\Sigma}^{t} \mathbf{N} d A \tag{2.26}
\end{equation*}
$$

then $\boldsymbol{\Sigma}$ denotes the "First Piola-Kirchhoff" stress tensor with respect to the undeformed surface area.

From Nanson's formula $\mathbf{n} d a=J \mathbf{F}^{-t} \mathbf{N} d A$ we get

$$
\begin{equation*}
J \boldsymbol{\sigma}^{t} \mathbf{F}^{-t} \mathbf{N} d A=\boldsymbol{\Sigma}^{t} \mathbf{N} d A \tag{2.27}
\end{equation*}
$$

or

$$
\begin{equation*}
\boldsymbol{\Sigma}=J \mathbf{F}^{-1} \boldsymbol{\sigma} \tag{2.28}
\end{equation*}
$$

which shows the loss of symmetry of the first Piola-Kirchhoff stress, $\boldsymbol{\Sigma} \neq \boldsymbol{\Sigma}^{t}$.

The "Second Piola-Kirchoff" stress is defined as

$$
\begin{equation*}
\mathbf{S}=\boldsymbol{\Sigma} \mathbf{F}^{-t}=J \mathbf{F}^{-1} \boldsymbol{\sigma} \mathbf{F}^{-t} \tag{2.29}
\end{equation*}
$$

or

$$
\begin{equation*}
\mathbf{S}=\mathbf{F}^{-1} \boldsymbol{\tau} \mathbf{F}^{-t} \tag{2.30}
\end{equation*}
$$

in which $\boldsymbol{\tau}=J \boldsymbol{\sigma}$ denotes the Kirchoff stress. From $\boldsymbol{\sigma}=\frac{1}{J} \mathbf{F} \mathbf{S F}^{t}$, we can relate the Kirchhoff stress to the second Piola-Kirchhoff stress

$$
\begin{equation*}
\boldsymbol{\tau}=\mathbf{F} \mathbf{S} \mathbf{F}^{t} \tag{2.31}
\end{equation*}
$$

## Balance of Linear Momentum:

In the current reference configuration

$$
\begin{equation*}
\frac{D}{D t}\left(\int_{\mathcal{B}} \rho \dot{\mathbf{x}} d v\right)=\int_{\mathcal{B}} \mathbf{b} d v+\int_{\partial \mathcal{B}} \mathbf{t}_{n} d a \tag{2.32}
\end{equation*}
$$

in which $\mathbf{t}_{n}=\boldsymbol{\sigma}^{t} \cdot \mathbf{n}$. In the initial undeformed reference configuration this reads

$$
\begin{equation*}
\int_{\mathcal{B}_{0}} \rho_{0} \frac{D \dot{\mathbf{x}}}{D t} d V=\int_{\mathcal{B}_{0}} \mathbf{b}_{0} d V+\int_{\partial \mathcal{B}_{0}} \boldsymbol{\Sigma}^{t} \mathbf{N} d A \tag{2.33}
\end{equation*}
$$

### 2.5 Mechanical Stress Power:

Conjugate forms of kinematic and static measures lead to an inner product form of stress and deformation rate. Using the divergence theorem the spatial description of momentum balance leads to the local statement of differential equilibrium :

$$
\begin{equation*}
\operatorname{div} \boldsymbol{\sigma}^{t}+\mathbf{b}=\rho \frac{D \mathbf{v}}{D t} \tag{2.34}
\end{equation*}
$$

If this equation of motion is scalar multiplied with $\mathbf{v}$ and integrated over the entire body, we get

$$
\begin{equation*}
\int_{\mathcal{B}}\left(d i v \boldsymbol{\sigma}^{t}\right) \cdot \mathbf{v} d v+\int_{\mathcal{B}} \mathbf{b} \cdot \mathbf{v} d v=\int_{\mathcal{B}} \rho \dot{\mathbf{v}} \cdot \mathbf{v} d v \tag{2.35}
\end{equation*}
$$

With $\left(\operatorname{div} \boldsymbol{\sigma}^{t}\right) \cdot \mathbf{v}=\operatorname{div}\left(\boldsymbol{\sigma}^{t} \cdot \mathbf{v}\right)-\boldsymbol{\sigma}: \mathbf{d}$, we get

$$
\begin{equation*}
\int_{\mathcal{B}} \mathbf{b} \cdot \mathbf{v} d v+\int_{\partial \mathcal{B}} \mathbf{t} \cdot \mathbf{v} d a=\int_{\mathcal{B}} \boldsymbol{\sigma}: \mathbf{d} d v+\frac{D}{D t} \int_{\mathcal{B}} \rho \dot{\mathbf{v}} . \mathbf{v} d v \tag{2.36}
\end{equation*}
$$

Using the divergence theorem the material description of momentum balance leads to the local statement of differential equilibrium :

$$
\begin{equation*}
\operatorname{Div} \boldsymbol{\Sigma}^{t}+\mathbf{b}_{0}=\rho_{0} \frac{D \mathbf{V}}{D t} \tag{2.37}
\end{equation*}
$$

If this equation is multiplied with the weighing function $\mathbf{v}$ and integrated over the entire body in the reference configuration, we find

$$
\begin{equation*}
\int_{\mathcal{B}_{0}}\left(D i v \boldsymbol{\Sigma}^{t}\right) \cdot \mathbf{v} d V+\int_{\mathcal{B}_{0}} \mathbf{b}_{\mathbf{0}} \cdot \mathbf{v} d V=\int_{\mathcal{B}_{0}} \rho_{0} \frac{D}{D t} \dot{\mathbf{V}} \cdot \mathbf{v} d V \tag{2.38}
\end{equation*}
$$

After analogous calculation to the spatial description we get

$$
\begin{equation*}
\int_{\mathcal{B}_{0}} \mathbf{b}_{0} \cdot \mathbf{v} d V+\int_{\partial \mathcal{B}_{0}}\left(\boldsymbol{\Sigma}^{t} \cdot \mathbf{N}\right) \cdot \mathbf{v} d A=\int_{\mathcal{B}_{0}} \boldsymbol{\Sigma}^{t}: \dot{\mathbf{F}} d V+\int_{\mathcal{B}_{0}} \rho_{0} \dot{\mathbf{V}} \cdot \mathbf{v} d V \tag{2.39}
\end{equation*}
$$

## Stress Power:

Considering $\boldsymbol{\tau}=\rho \boldsymbol{\sigma}$, the inner product of stress and the rate of deformation leads to alternative representations in terms of conjugate values

$$
\begin{equation*}
\dot{W}_{\sigma}=\frac{1}{\rho_{0}} \boldsymbol{\tau}: \mathbf{d}=\frac{1}{\rho} \boldsymbol{\sigma}: \mathbf{d}=\frac{1}{\rho_{0}} \boldsymbol{\Sigma}^{t}: \dot{\mathbf{F}}=\frac{1}{\rho_{0}} \mathbf{S}: \dot{\mathbf{E}}_{G} \tag{2.40}
\end{equation*}
$$

## Chapter 3

## Kinematics of Continua


#### Abstract

In this Chapter, the motion and deformation of continua will be reviewed:


- Kinematics of Motion: X, x, u.
- Kinematics of Deformation: F, E, e.


### 3.1 Kinematics of Motion:

The motion of continua may be described in two ways:
(a) In the Lagrangean description the continuous medium is considered to be comprised of particles the motion of which is of primary interest. The Lagrangean coordinates describe the spatial variation of a field variable in terms of the particle position $\mathbf{X}$ in the initial reference configuration.
(b) In the Eulerian description the location is of primary interest which is occupied by particles at the time t. The Eulerian coordinates describe the spatial variation of a field variable in terms of the spatial domain $\mathbf{x}$ occupied by the continuum.

Usually, Eulerian coordinates are used to study the motion of fluids through a fixed spatial domain, while Lagrangean coordinates are primarily used to follow the particle motion of solids. In sum, every function expressed in Lagrangean coordinates may be transformed into Eulerian coordinates and vice-versa.

- In Lagrangean coordinates, which are known as material coordinates, the initial position of a particle is used to label the material particle under consideration:
- Lagrange Coordinates: (Material Description)

$$
\begin{equation*}
\mathbf{X}=X_{A} \mathbf{e}_{A} \quad \text { where } \quad A=1,2,3 \tag{3.1}
\end{equation*}
$$

where $\mathbf{e}_{A}$ denote the orthogonal material base vectors.

- In Eulerian coordinates, which are known as spatial coordinates, the location is used to label the material position of a material particle at the time $t$ :
- Euler Coordinates: (Spatial Description)

$$
\begin{equation*}
\mathbf{x}=x_{i} \mathbf{e}_{i} \quad \text { where } \quad i=1,2,3 \tag{3.2}
\end{equation*}
$$

where $\mathbf{e}_{i}$ denote the orthogonal spatial base vectors.

The scalar temperature field may be represented by :

- Lagrangean Coordinates: $T=T(\mathbf{X}, t)$ - material description.
- Eulerian Coordinates : $T=T(\mathbf{x}, t)$ - spatial description.

In Lagrangean coordinates the temperature at every material point is studied, while in Eulerian coordinates the temperature at a fixed location which the material occupies is of primary interest.

During the motion, the deformation gradient is defined as:

$$
\begin{equation*}
\mathbf{F}=\frac{\partial \mathbf{x}}{\partial \mathbf{X}} \Rightarrow d \mathbf{x}=\mathbf{F} \cdot d \mathbf{X} \tag{3.3}
\end{equation*}
$$

where $\mathbf{X}$ denotes the position of a arbitrary material particle in the (initial) reference configuration, and $\mathbf{x}$ its position in the current configuration. The determinant $\operatorname{det} \mathbf{F}$ is known as the Jacobian of the deformation gradient. Therefore the mapping of the material line element $d \mathbf{X}$ from the reference into the current configuration is defined by :

$$
\left[\begin{array}{c}
d x_{i}  \tag{3.4}\\
d x_{j} \\
d x_{k}
\end{array}\right]=\left[\begin{array}{lll}
\frac{d x_{i}}{d X_{A}} & \frac{d x_{i}}{d X_{B}} & \frac{d x_{i}}{d X_{C}} \\
\frac{d x_{j}}{d X_{A}} & \frac{d x_{j}}{d X_{B}} & \frac{d x_{j}}{d X_{C}} \\
\frac{d x_{k}}{d X_{A}} & \frac{d x_{k}}{d X_{B}} & \frac{d x_{k}}{d X_{C}}
\end{array}\right]\left[\begin{array}{l}
d X_{A} \\
d X_{B} \\
d X_{C}
\end{array}\right]
$$

Restriction: it can be shown that the Jacobian $J=\operatorname{det} \mathbf{F} \neq 0$ defines the volume change.

In other terms, for rigid body motions, $J=1$ in the absence of deformation. For a unique one-to-one defomation map $\operatorname{det} \mathbf{F} \neq 0$ must hold.

### 3.2 Polar Decomposition:

The polar decomposition theorem states that the deformation gradient $\mathbf{F}$ may be decomposed uniquely into a positive definite tensor and a proper orthogonal tensor, i.e. the right $\mathbf{U}$ or left $\mathbf{V}$ stretch tensor plus the rotation $\mathbf{R}$ tensor.

1. Right Polar Decomposition: $\mathbf{F}=\mathbf{R} \cdot \mathbf{U}$
where: $\quad \operatorname{det} \mathbf{R}=1, \quad \mathbf{R}^{t} \cdot \mathbf{R}=\mathbf{1}, \quad \mathbf{R} \cdot \mathbf{R}^{t}=\mathbf{1}$,
Note, $\mathbf{U}=\mathbf{U}^{t}$ such that $\lambda_{U}>0$ and

$$
\begin{equation*}
\mathbf{U}^{2}=\mathbf{U}^{t} \cdot \mathbf{R}^{t} \cdot \mathbf{R} \cdot \mathbf{U}=\mathbf{F}^{t} \cdot \mathbf{F} \tag{3.5}
\end{equation*}
$$

2. Left Polar Decomposition: $\mathbf{F = V} \cdot \mathbf{R}$
where:

$$
\begin{equation*}
\mathbf{V}^{2}=\mathbf{F} \cdot \mathbf{F}^{t}=\mathbf{V} \cdot \mathbf{R} \cdot \mathbf{R}^{t} \cdot \mathbf{V} \tag{3.6}
\end{equation*}
$$

Note $\mathbf{V}=\mathbf{V}^{t}$ such that $\lambda_{V}>0$ and $\lambda_{V}=\lambda_{U}$

The Physical Meaning of $\mathbf{U}$ and $\mathbf{V}$ is:

- Right Decomposition: $d \mathbf{x}=\mathbf{R} \cdot \mathbf{U} \cdot d \mathbf{X}$, which means that $d \mathbf{X}$ is first stretched and then rotated.
- Left Decomposition: $d \mathbf{x}=\mathbf{V} \cdot \mathbf{R} \cdot d \mathbf{X}$, which means that $d \mathbf{X}$ is first rotated and then stretched.

If $\mathbf{F}$ is non-singular $\Rightarrow \operatorname{det} \mathbf{F} \neq 0$ and positive, then there exists a unique decomposition into a proper orthogonal tensor $\mathbf{R}$ and a positive definite tensor $\mathbf{U}$ or $\mathbf{V}$.

### 3.3 Strain Definition:

In this section alternative strain measures are introduced to define strain.
Extensional Engineering Strain:
In the uniaxial case the engineering strain is simply the length change normalized by the original length:

$$
\begin{equation*}
\epsilon_{e n g}=\frac{\Delta L}{L_{0}}=\frac{L-L_{0}}{L_{0}} . \tag{3.7}
\end{equation*}
$$

The triaxial extension of the engineering strain will be dicussed later on in the context of infinitesimal deformations.

Extensional Logarithmic Strain [Hencky, 1928]:
In the uniaxial case the logarithmic strain is defined by integrating the stretch rate:

$$
\begin{equation*}
\epsilon_{l n}=\int_{L_{0}}^{L} \frac{d \ell}{\ell}=\ln \frac{L}{L_{0}}=\ln \lambda \quad \text { where } \quad \lambda=\frac{L}{L_{0}} . \tag{3.8}
\end{equation*}
$$

The Triaxial extension of the logarithmic strain may be described in two ways:

- Lagrangean format : $\boldsymbol{\epsilon}_{l n}^{U}=\ln \mathbf{U}$, with principal coordinates which are defined by $\mathbf{e}_{U}$
- Eulerian format : $\boldsymbol{\epsilon}_{l n}^{V}=\ln \mathbf{V}$, with principal Coordinates which are defined by $\mathbf{e}_{V}$ where the two base triads are related by the rotation $\mathbf{e}_{V}=\mathbf{R} \cdot \mathbf{e}_{U}$.

Spectral Representation :

$$
\begin{align*}
\mathbf{U} & =\sum_{i=1}^{3} \lambda_{i} \mathbf{e}_{U}^{i} \otimes \mathbf{e}_{U}^{i} \quad \text { and } \quad \boldsymbol{\epsilon}_{l n}^{U}=\sum_{i=1}^{3}\left(\ln \lambda_{i}\right) \mathbf{e}_{U}^{i} \otimes \mathbf{e}_{U}^{i}  \tag{3.9}\\
\mathbf{V} & =\sum_{i=1}^{3} \lambda_{i} \mathbf{e}_{V}^{i} \otimes \mathbf{e}_{V}^{i} \quad \text { and } \quad \boldsymbol{\epsilon}_{l n}^{V}=\sum_{i=1}^{3}\left(\ln \lambda_{i}\right) \mathbf{e}_{V}^{i} \otimes \mathbf{e}_{V}^{i} \tag{3.10}
\end{align*}
$$

### 3.3.1 Lagrangean Strain Measures:

If $d S$ is the infinitesimal line element measuring the initial distance of two points, and if $d s$ is the current length of the line element between those two points, then one can define the extensional deformation is defined as:

$$
\begin{equation*}
d s^{2}-d S^{2}=d \mathbf{x}^{t} \cdot d \mathbf{x}-d \mathbf{X}^{t} \cdot d \mathbf{X}=d \mathbf{X}^{t} \cdot\left[\mathbf{F}^{t} \cdot \mathbf{F}-\mathbf{1}\right] \cdot d \mathbf{X} \tag{3.11}
\end{equation*}
$$

The term in bracket defines the Green strain which is related to the right stretch tensor by:

$$
\begin{equation*}
\mathbf{E}_{G}=\frac{1}{2}\left[\mathbf{F}^{t} \cdot \mathbf{F}-\mathbf{1}\right]=\frac{1}{2}\left[\mathbf{U}^{2}-\mathbf{1}\right] \tag{3.12}
\end{equation*}
$$

In indicial form,

$$
\begin{gather*}
\mathbf{F}=\frac{\partial x_{i}}{\partial X_{A}} \mathbf{e}_{i} \otimes \mathbf{e}_{A}  \tag{3.13}\\
\mathbf{U}^{2}=\frac{\partial x_{i}}{\partial X_{A}} \cdot \frac{\partial x_{i}}{\partial X_{B}}  \tag{3.14}\\
E_{A B}^{G}=\frac{1}{2}\left(\frac{\partial x_{i}}{\partial X_{A}} \cdot \frac{\partial x_{i}}{\partial X_{B}}-\delta_{A B}\right) \tag{3.15}
\end{gather*}
$$

Generalized Lagrangean Strain [Doyle-Erickson 1956]:
The generalized format of Lagrangean strain is defined as:

$$
\begin{equation*}
\mathbf{E}_{m}=\frac{1}{m}\left[\mathbf{U}^{m}-\mathbf{1}\right] \tag{3.16}
\end{equation*}
$$

where $m$ is an integer, where
$\mathrm{m}=0$, defines the logarithmic strain:

$$
\begin{equation*}
\mathbf{E}_{0}=\ln \mathbf{U} \tag{3.17}
\end{equation*}
$$

$\mathrm{m}=1$, defines the Biot strain:

$$
\begin{equation*}
\mathbf{E}_{1}=[\mathbf{U}-\mathbf{1}] \tag{3.18}
\end{equation*}
$$

$\mathrm{m}=2$, defines the Green strain:

$$
\begin{equation*}
\mathbf{E}_{2}=\frac{1}{2}\left[\mathbf{U}^{2}-\mathbf{1}\right] \tag{3.19}
\end{equation*}
$$

Using the definition of stretch $\lambda=\frac{L}{L_{0}}$, then 1-dim examples of extensional strain measures include :

$$
\begin{align*}
& E_{0}=\ln \lambda=\ln \frac{L}{L_{0}} \\
& E_{1}=\lambda-1=\frac{L-L_{0}}{L_{0}}  \tag{3.20}\\
& E_{2}=\frac{1}{2}\left[\lambda^{2}-1\right]=\frac{1}{2}\left[\frac{L^{2}-L_{0}^{2}}{L_{0}^{2}}\right]
\end{align*}
$$

We see that $E_{0}$ reduces to the logarithmic Hencky strain, $E_{1}$ to the engineering strain , and $E_{2}$ to the Green strain.

## Strain-Displacement Relationship

If $\mathbf{X}$ and $\mathbf{x}$ are the initial and current position vectors of a particle, then the displacement vector defines the relation between $\mathbf{X}$ and $\mathbf{x}$ as:

$$
\begin{equation*}
\mathbf{x}=\mathbf{x}(\mathbf{X}, t)=\mathbf{u}(\mathbf{X}, t)+\mathbf{X} \tag{3.21}
\end{equation*}
$$

The deformation gradient is related to the displacement gradient,

$$
\begin{equation*}
\frac{\partial \mathbf{x}}{\partial \mathbf{X}}=\frac{\partial \mathbf{u}}{\partial \mathbf{X}}+\mathbf{1}=\mathbf{H}+\mathbf{1} \quad \text { where } \quad \frac{\partial \mathbf{u}}{\partial \mathbf{X}}=\mathbf{H} \tag{3.22}
\end{equation*}
$$

With $\mathbf{F}=\mathbf{H}+\mathbf{1}$ the material stretch tensor reads,

$$
\begin{equation*}
\mathbf{U}^{2}=\mathbf{F}^{t} \cdot \mathbf{F}=\mathbf{1}+\mathbf{H}+\mathbf{H}^{t}+\mathbf{H}^{t} \cdot \mathbf{H} \tag{3.23}
\end{equation*}
$$

and the Lagrangean strain-displacement relationship defines the Green strain in terms of:

$$
\begin{equation*}
\mathbf{E}_{G}=\frac{1}{2}\left[\mathbf{1}+\mathbf{H}+\mathbf{H}^{t}+\mathbf{H}^{t} \cdot \mathbf{H}\right] \tag{3.24}
\end{equation*}
$$

### 3.3.2 Eulerian Strain Measures :

Parallel to the material description, the extensional deformation may be defined in terms of spatial coordinates,

$$
\begin{equation*}
d s^{2}-d S^{2}=d \mathbf{x}^{t} \cdot\left[\mathbf{1}-\mathbf{F}^{-t} \cdot \mathbf{F}^{-1}\right] \cdot d \mathbf{x} \tag{3.25}
\end{equation*}
$$

The term in the bracket defines the Almansi strain which is related to the left stretch tensor by,

$$
\begin{equation*}
\mathbf{e}_{A}=\frac{1}{2}\left[\mathbf{1}-\mathbf{F}^{-t} \cdot \mathbf{F}^{-1}\right]=\frac{1}{2}\left[\mathbf{1}-\left(\mathbf{V}^{2}\right)^{-1}\right] \tag{3.26}
\end{equation*}
$$

Generalized Eulerian Strain [Doyle-Erickson 1956]:

The generalized Eulerian strain tensor is defined as:

$$
\begin{equation*}
\mathbf{e}_{m}=\frac{1}{m}\left(\mathbf{V}^{m}-\mathbf{1}\right) \tag{3.27}
\end{equation*}
$$

where $m$ is an integer.
$\mathrm{m}=0$, defines the spatial format of the logarithmic strain:

$$
\begin{equation*}
\mathbf{e}_{0}=\ln \mathbf{V} \tag{3.28}
\end{equation*}
$$

$\mathrm{m}=-1$, defines the spatial format of the Biot strain

$$
\begin{equation*}
\mathbf{e}_{-1}=\left[\mathbf{1}-\mathbf{V}^{-1}\right] \tag{3.29}
\end{equation*}
$$

$\mathrm{m}=-2$, defines the Almansi strain

$$
\begin{equation*}
\mathbf{e}_{-2}=\frac{1}{2}\left[\mathbf{1}-\mathbf{V}^{-2}\right] \tag{3.30}
\end{equation*}
$$

using the inverse stretch measure $\frac{1}{\lambda}=\frac{L_{0}}{L}$, then 1-dim examples include:

$$
\begin{align*}
e_{0} & =-\ln \frac{1}{\lambda}=\ln \frac{L}{L} \\
e_{-1} & =1-\frac{1}{\lambda}=\frac{L-L_{0}}{L}  \tag{3.31}\\
e_{-2} & =\frac{1}{2}\left[1-\frac{1}{\lambda^{2}}\right]=\frac{1}{2}\left[\frac{L^{2}-L_{0}^{2}}{L^{2}}\right]
\end{align*}
$$

We see that in 1-dim $e_{0}$ coincides with the logarithmic Hencky strain, $e_{1}$ corresponds to the engineering strain in which the length change is however normalized by the current length, while the Almansi strain $e_{-2}$ corresponds to the Green strain $E_{2}$.

## Strain-Displacement Relationship

In the spatial description the motion is described by the inverse relation:

$$
\begin{equation*}
\mathbf{u}(\mathbf{x}, t)=\mathbf{x}-\mathbf{X}(\mathbf{x}, t) \tag{3.32}
\end{equation*}
$$

The spatial displacement gradient is defined as:

$$
\begin{equation*}
\mathbf{h}=\frac{\partial \mathbf{u}}{\partial \mathbf{x}}=\mathbf{1}-\frac{\partial \mathbf{X}}{\partial \mathbf{x}}=\mathbf{1}-\mathbf{F}^{-1} \tag{3.33}
\end{equation*}
$$

or

$$
\begin{align*}
\mathbf{F}^{-1} & =\mathbf{1}-\mathbf{h}  \tag{3.34}\\
\mathbf{V}^{-2}=\mathbf{F}^{-t} \cdot \mathbf{F}^{-1} & =\mathbf{1}-\mathbf{h}-\mathbf{h}^{t}+\mathbf{h}^{t} \cdot \mathbf{h} \tag{3.35}
\end{align*}
$$

Substituting into the expression of the Almansi strain $\mathbf{e}_{A}$ renders the Eulerian strain-displacement relationship in the form,

$$
\begin{equation*}
\mathbf{e}_{A}=\frac{1}{2}\left[\mathbf{h}+\mathbf{h}^{t}-\mathbf{h}^{t} \cdot \mathbf{h}\right] \tag{3.36}
\end{equation*}
$$

### 3.3.3 Infinitesimal Deformations and Rotations:

If the displacement gradient $\mathbf{H}$ is very small, then $\operatorname{det} \mathbf{H} \ll 1 \Rightarrow \operatorname{det}\left(\mathbf{H}^{t} \cdot \mathbf{H}\right) \simeq 0$. Thus,

$$
\begin{equation*}
\boldsymbol{\epsilon}=\frac{1}{2}\left[\mathbf{H}+\mathbf{H}^{t}\right] \simeq \frac{1}{2}\left[\mathbf{h}+\mathbf{h}^{t}\right] \tag{3.37}
\end{equation*}
$$

In short, in the case of "Infinitesimal Deformations" the spatial and the material displacement gradients coincide,

$$
\begin{equation*}
\frac{\partial \mathbf{u}}{\partial \mathbf{X}} \sim \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \tag{3.38}
\end{equation*}
$$

Additive decomposition into symmetric and skew-symmetric components leads to

$$
\begin{equation*}
\mathbf{H}=\frac{\partial \mathbf{u}}{\partial \mathbf{X}}=\frac{1}{2}\left[\mathbf{H}+\mathbf{H}^{t}\right]+\frac{1}{2}\left[\mathbf{H}-\mathbf{H}^{t}\right] \tag{3.39}
\end{equation*}
$$

where the symmetric part defines the traditional linearized strain tensor

$$
\begin{equation*}
\epsilon_{i j}=\frac{1}{2}\left[\frac{\partial u_{i}}{\partial X_{j}}+\frac{\partial u_{j}}{\partial X_{i}}\right] \tag{3.40}
\end{equation*}
$$

and where the skew-symmetric part defines the infinitesimal rotation tensor

$$
\begin{equation*}
\omega_{i j}=\frac{1}{2}\left[\frac{\partial u_{i}}{\partial X_{j}}-\frac{\partial u_{j}}{\partial X_{i}}\right] \tag{3.41}
\end{equation*}
$$

such that $\epsilon_{i j}=\epsilon_{j i}$ and $\omega_{i j}=-\omega_{j i}$.

## Chapter 4

## Elastic Material Models

Linear elasticity is the main staple of material models in solids and structures. The statement 'ut tensio sic vis' attributed to Robert Hooke (1635-1703) characterizes the behavior of a linear spring in which the deformations increase proportionally with the applied forces according to the anagram 'ceiiinossstuv'. The original format of Hooke's law included the geometric properties of the wire test specimens, and therefore the spring constant did exhibit a pronounced size effect. The definition of the modulus of elasticity $E$, where

$$
\begin{equation*}
\sigma=E \epsilon \tag{4.1}
\end{equation*}
$$

is attributed to Thomas Young (1773-1829). He expressed the proportional material behavior through the notion of a normalized force density and a normalized deformation measure, though the original formulation also did not entirely eliminate the size effect.
The tensorial character of stress was established by Cauchy, who defined the triaxial state of stress by three traction vectors using the celebrated tetraeder argument of equilibrium. The state of stress is described in terms of Cartesian coordinates by the second order tensor

$$
\boldsymbol{\sigma}(\boldsymbol{X}, t)=\left[\begin{array}{lll}
\sigma_{11} & \sigma_{12} & \sigma_{13}  \tag{4.2}\\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{array}\right]
$$

The conjugate state of strain is a second order tensor with Cartesian coordinates,

$$
\boldsymbol{\epsilon}(\boldsymbol{X}, t)=\left[\begin{array}{ccc}
\epsilon_{11} & \epsilon_{12} & \epsilon_{13}  \tag{4.3}\\
\epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\
\epsilon_{31} & \epsilon_{32} & \epsilon_{33}
\end{array}\right]
$$

which is normally expressed in terms of the symmetric part of the displacement gradient, if we restrict our attention to infinitesimal deformations. In the case of non-polar media we may confine our attention to stress measures, which are symmetric according to the axiom of L. Boltzmann,

$$
\begin{equation*}
\boldsymbol{\sigma}=\boldsymbol{\sigma}^{t} \quad \text { or } \quad \sigma_{i j}=\sigma_{j i} \tag{4.4}
\end{equation*}
$$

and the conjugate strain measures

$$
\begin{equation*}
\boldsymbol{\epsilon}=\boldsymbol{\epsilon}^{t} \quad \text { or } \quad \epsilon_{i j}=\epsilon_{j i} \tag{4.5}
\end{equation*}
$$

where $i=1,2,3$ and $j=1,2,3$. As a result, the eigenvalues are real-valued and constitute the set of principal stresses and strains with zero shear components in the principal eigen-directions of the second order tensor. In contrast, non-symmetric stress and strain measures may exhibit complex conjugate principal values and maximuum normal stress and strain components in directions with non-zero shear components characteristic for micropolar Cosserat continua.
Restricting this exposition to symmetric stress and strain tensors they may be cast into vector form using the Voigt notation of crystal physics.

$$
\boldsymbol{\sigma}(\boldsymbol{X}, t)=\left[\begin{array}{llllll}
\sigma_{11} & \sigma_{22} & \sigma_{33} & \tau_{12} & \tau_{23} & \tau_{31} \tag{4.6}
\end{array}\right]^{t}
$$

and

$$
\boldsymbol{\epsilon}(\boldsymbol{X}, t)=\left[\begin{array}{llllll}
\epsilon_{11} & \epsilon_{22} & \epsilon_{33} & \gamma_{12} & \gamma_{23} & \gamma_{31} \tag{4.7}
\end{array}\right]^{t}
$$

where $\tau_{i j}=\sigma_{i j}, \gamma_{i j}=2 \epsilon_{i j}, \forall i \neq j$. The vector form of stress and strain will allow us to formulate material models in matrix notation used predominantly in engineering, (some of the properties of second order tensors and basic tensor operations are expanded in Appendix I).

### 4.1 Linear Elastic Material Behavior:

Generalization of the scalar format of Hooke's law is based on the notion that the triaxial state of stress is proportional to the triaxial state of strain through the linear transformation,

$$
\begin{equation*}
\boldsymbol{\sigma}=\mathcal{E}: \boldsymbol{\epsilon} \quad \text { or } \quad \sigma_{i j}=\mathcal{E}_{i j k l} \epsilon_{k l} \tag{4.8}
\end{equation*}
$$

Considering the symmetry of the stress and strain, the elasticity tensor involves in general 36 elastic moduli. This may be further reduced to 21 elastic constants, if we invoke major symmetry of the elasticity tensor, i.e.

$$
\begin{equation*}
\mathcal{E}=\mathcal{E}^{t} \quad \text { or } \quad \mathcal{E}_{i j k l}=\mathcal{E}_{k l i j} \quad \text { with } \quad \mathcal{E}_{i j k l}=\mathcal{E}_{i j l k} \quad \text { and } \quad \mathcal{E}_{i j k l}=\mathcal{E}_{j i k l} \tag{4.9}
\end{equation*}
$$

The task of identifying 21 elastic moduli is simplified if we consider specific classes of symmetry, whereby orthotropic elasticity involves nine, and transversely anisotropic elasticity five elastic moduli.

## 1. Isotropic Linear Elasticity

In the case of isotropy the fourth order elasticity tensor has the most general representation,

$$
\begin{equation*}
\mathcal{E}=a_{o} \mathbf{1} \otimes \mathbf{1}+a_{1} \mathbf{1} \bar{\otimes} \mathbf{1}+a_{2} \mathbf{1} \underline{\otimes} \mathbf{1} \quad \text { or } \quad \mathcal{E}_{i j k l}=a_{o} \delta_{i j} \delta_{k l}+a_{1} \delta_{i k} \delta_{j l}+a_{2} \delta_{i l} \delta_{j k} \tag{4.10}
\end{equation*}
$$

where $\mathbf{1}=\left[\delta_{i j}\right]$ stands for the second order unit tensor. The three parameter expression may be recast in terms of symmetric and skew symmetric fourth order tensor components as

$$
\begin{equation*}
\mathcal{E}=a_{o} \mathbf{1} \otimes \mathbf{1}+b_{1} \mathcal{I}+b_{2} \mathcal{I}^{\text {skew }} \tag{4.11}
\end{equation*}
$$

where the symmetric fourth order unit tensor reads

$$
\begin{equation*}
\boldsymbol{I}=\frac{1}{2}[\mathbf{1} \bar{\otimes} \mathbf{1}+\mathbf{1} \otimes \mathbf{1}] \quad \text { or } \quad \mathcal{I}_{i j k l}=\frac{1}{2}\left[\delta_{i k} \delta_{j l}+\delta_{i l} \delta_{j k}\right] \tag{4.12}
\end{equation*}
$$

and the skewed symmetric one

$$
\begin{equation*}
\boldsymbol{\mathcal { I }}^{\text {skew }}=\frac{1}{2}[\mathbf{1} \bar{\otimes} \mathbf{1}-\mathbf{1} \otimes \mathbf{1}] \quad \text { or } \quad \mathcal{I}_{i j k l}^{s k e w}=\frac{1}{2}\left[\delta_{i k} \delta_{j l}-\delta_{i l} \delta_{j k}\right] \tag{4.13}
\end{equation*}
$$

Because of the symmetry of stress and strain the skewed symmetric contribution is inactive, $b_{2}=0$, thus isotropic linear elasticity the material behavior is fully described by two independent elastic constants. In short, the fourth order material stiffness tensor reduces to

$$
\begin{equation*}
\mathcal{E}=\Lambda \mathbf{1} \otimes \mathbf{1}+2 G \mathcal{I} \quad \text { or } \quad \mathcal{E}_{i j k l}=\Lambda \delta_{i j} \delta_{k l}+G\left[\delta_{i k} \delta_{j l}+\delta_{i l} \delta_{j k}\right] \tag{4.14}
\end{equation*}
$$

where the two elastic constants $\Lambda, G$ are named after Gabriel Lamé (1795-1870).

$$
\begin{equation*}
\Lambda=\frac{E \nu}{[1+\nu][1-2 \nu]} \tag{4.15}
\end{equation*}
$$

denotes the cross modulus, and

$$
\begin{equation*}
G=\frac{E}{2[1+\nu]} \tag{4.16}
\end{equation*}
$$

designates the shear modulus which have a one-to-one relationship with the modulus of elasticity and Poisson's ratio, $E, \nu$.
In the absence of initial stresses and initial strains due to environmental effects, the linear elastic relation reduces to

$$
\begin{equation*}
\boldsymbol{\sigma}=\Lambda[\operatorname{tr} \boldsymbol{\epsilon}] \mathbf{1}+2 G \boldsymbol{\epsilon} \quad \text { or } \quad \sigma_{i j}=\Lambda \epsilon_{k k} \delta_{i j}+2 G \epsilon_{i j} \tag{4.17}
\end{equation*}
$$

Here the trace operation is the sum of the diagonal entries of the second order tensor corresponding to double contraction with the identity tensor $\operatorname{tr} \boldsymbol{\epsilon}=\epsilon_{k k}=\mathbf{1}: \boldsymbol{\epsilon}$.
2. Matrix Form of Elastic Stiffness: $\boldsymbol{\sigma}=\boldsymbol{E} \boldsymbol{\epsilon}$

Isotropic linear elastic behavior may be cast in matrix format, using the Voigt notation of symmetric stress and strain tensors and the engineering definition of shear strain $\gamma_{i j}=2 \epsilon_{i j}$. The elastic stiffness matrix may be written for isotropic behavior as,

$$
\boldsymbol{E}=\left[\begin{array}{ccc|ccc}
\Lambda+2 G & \Lambda & \Lambda & &  \tag{4.18}\\
\Lambda & \Lambda+2 G & \Lambda & & 0 & \\
\Lambda & \Lambda & \Lambda+2 G & & & \\
\hline & & & G & & \\
& 0 & & & G & \\
& & & &
\end{array}\right]
$$

3. Matrix Form of Elastic Compliance: $\boldsymbol{\epsilon}=\boldsymbol{C} \boldsymbol{\sigma}$

In the isotropic case the normal stress $\sigma_{11}$ gives rise to three normal strain contributions, the direct strain $\epsilon_{11}=\frac{1}{E} \sigma_{11}$ and the normal strains $\epsilon_{22}=-\frac{\nu}{E} \sigma_{11}, \epsilon_{33}=\frac{\nu}{E} \sigma_{11}$ because of the cross effect attributed to Siméon Denis Poisson (1781-1840). Using the principle of superposition, the additional strain contributions due to $\sigma_{22}$ and $\sigma_{33}$ enter the compliance relation for isotropic elasticity in matrix format,

$$
\left[\begin{array}{l}
\epsilon_{11}  \tag{4.19}\\
\epsilon_{22} \\
\epsilon_{33}
\end{array}\right]=\frac{1}{E}\left[\begin{array}{ccc}
1 & -\nu & -\nu \\
-\nu & 1 & -\nu \\
-\nu & -\nu & 1
\end{array}\right]\left[\begin{array}{l}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33}
\end{array}\right]
$$

In the isotropic case the shear response is entirely decoupled from the direct response of the normal components. Thus the compliance matrix expands into the partitioned form

$$
\boldsymbol{C}=\frac{1}{E}\left[\begin{array}{ccc|cc}
1 & -\nu & -\nu & &  \tag{4.20}\\
-\nu & 1 & -\nu & 0 & \\
-\nu & -\nu & 1 & & \\
\hline & 0 & & 2[1+\nu] & \\
& & 2[1+\nu] & \\
& & & & 2[1+\nu]
\end{array}\right]
$$

where isotropy entirely decouples the shear response from the normal stress-strain response. This cross effect of Poisson is illustrated in Figure 2, which shows the interaction of lateral and axial deformations under axial compression. It is intriguing that in his original work a value of $\nu=0.25$ was proposed by S. Poisson based on molecular considerations. The elastic compliance relation reads in direct and index notations,

$$
\begin{equation*}
\mathcal{C}=-\frac{\nu}{E} \mathbf{1} \otimes \mathbf{1}+\frac{1}{2 G} \boldsymbol{\mathcal { I }} \quad \text { or } \quad C_{i j k l}=-\frac{\nu}{E} \delta_{i j} \delta_{k l}+\frac{1+\nu}{2 E}\left[\delta_{i k} \delta_{j l}+\delta_{i l} \delta_{j k}\right] \tag{4.21}
\end{equation*}
$$

## 4. Canonical Format of Isotropic Elasticity:

Decomposing the stress and strain tensors into spherical and deviatoric components

$$
\begin{array}{rlll}
\boldsymbol{s}=\boldsymbol{\sigma}-\sigma_{v o l} \mathbf{1} & \text { where } & \sigma_{v o l}=\frac{1}{3}[\operatorname{tr\boldsymbol {\sigma }]} \\
\boldsymbol{e}=\boldsymbol{\epsilon}-\epsilon_{v o l} \mathbf{1} & \text { where } & \epsilon_{v o l}=\frac{1}{3}[\operatorname{tr\boldsymbol {\epsilon }}] \tag{4.23}
\end{array}
$$

leads to the stress deviator

$$
\boldsymbol{s}(\boldsymbol{x}, t)=\left[\begin{array}{ccc}
\frac{1}{3}\left[2 \sigma_{11}-\sigma_{22}-\sigma_{33}\right] & \sigma_{12} & \sigma_{13}  \tag{4.24}\\
\sigma_{21} & \frac{1}{3}\left[2 \sigma_{22}-\sigma_{33}-\sigma_{11}\right] & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \frac{1}{3}\left[2 \sigma_{33}-\sigma_{11}-\sigma_{22}\right]
\end{array}\right]
$$

and the strain deviator

$$
\boldsymbol{e}(\boldsymbol{x}, t)=\left[\begin{array}{ccc}
\frac{1}{3}\left[2 \epsilon_{11}-\epsilon_{22}-\epsilon_{33}\right] & \epsilon_{12} & \epsilon_{13}  \tag{4.25}\\
\epsilon_{21} & \frac{1}{3}\left[2 \epsilon_{22}-\epsilon_{33}-\epsilon_{11}\right] & \epsilon_{23} \\
\epsilon_{31} & \epsilon_{32} & \frac{1}{3}\left[2 \epsilon_{33}-\epsilon_{11}-\epsilon_{22}\right]
\end{array}\right]
$$

which have the property trs $=0$ and tre $=0$. The decomposition decouples the volumetric from the distortional response, because of the underlying orthogonality of the spherical and deviatoric partitions, $s:\left[\sigma_{v o l} \mathbf{1}\right]=0$ and $e:\left[\epsilon_{v o l} \mathbf{1}\right]=0$. The decoupled response reduces the elasticity tensor to the scalar form,

$$
\begin{equation*}
\sigma_{v o l}=3 K \epsilon_{v o l} \quad \text { and } \quad s=2 G \boldsymbol{e} \tag{4.26}
\end{equation*}
$$

in which the bulk and the shear moduli,

$$
\begin{equation*}
K=\frac{E}{3[1-2 \nu]}=\Lambda+\frac{2}{3} G \quad \text { and } \quad G=\frac{E}{2[1+\nu]}=\frac{3}{2}[K-\Lambda] \tag{4.27}
\end{equation*}
$$

define the volumetric and the deviatoric material stiffness.
Consequently, the internal strain energy density expands into the canonical form of two decoupled contributions

$$
\begin{equation*}
2 W=\boldsymbol{\sigma}: \boldsymbol{\epsilon}=\left[\sigma_{v o l} \mathbf{1}\right]:\left[\epsilon_{v o l} \mathbf{1}\right]+\boldsymbol{s}: \boldsymbol{e}=9 K \epsilon_{v o l}^{2}+2 G \boldsymbol{e}: \boldsymbol{e} \tag{4.28}
\end{equation*}
$$

such that the positive strain energy argument delimits the range of possible values of Poisson's ratio to $-1 \leq \nu \leq 0.5$

### 4.1.1 Linear Isotropic Elasticity under Thermal Strain

In the case of isotropic material behavior, with no directional properties, the size of a representative volume element may change due to thermal effects or shrinkage and swelling, but it will not distort. Consequently, the expansion is purely volumetric, i.e. identical in all directions. Using direct and index notation, the additive decomposition of strain into elastic and initial volumetric components, $\boldsymbol{\epsilon}=\boldsymbol{\epsilon}_{e}+\boldsymbol{\epsilon}_{o}$, leads to the following extension of the elastic compliance relation:

$$
\begin{equation*}
\boldsymbol{\epsilon}=-\frac{\nu}{E}[\operatorname{tr} \boldsymbol{\sigma}] \mathbf{1}+\frac{1}{2 G} \boldsymbol{\sigma}+\epsilon_{o} \mathbf{1} \quad \text { or } \quad \epsilon_{i j}=-\frac{\nu}{E} \sigma_{k k} \delta_{i j}+\frac{1}{2 G} \sigma_{i j}+\epsilon_{o} \delta_{i j} \tag{4.29}
\end{equation*}
$$

where $\boldsymbol{\epsilon}_{o}=\alpha \Delta T \mathbf{1}$ denotes the initial volumetric strain e.g. due to thermal expansion when the temperature changes from the reference temperature, $\Delta T=T-T_{o}$. The inverse relation reads

$$
\begin{equation*}
\boldsymbol{\sigma}=\Lambda[t r \boldsymbol{\epsilon}] \mathbf{1}+2 G \boldsymbol{\epsilon}-3 \epsilon_{o} K \mathbf{1} \quad \text { or } \quad \sigma_{i j}=\Lambda \epsilon_{k k} \delta_{i j}+2 G \epsilon_{i j}-3 \epsilon_{o} K \delta_{i j} \tag{4.30}
\end{equation*}
$$

Rewriting this equation in matrix notation we have:

$$
\left[\begin{array}{l}
\sigma_{11}  \tag{4.31}\\
\sigma_{22} \\
\sigma_{33}
\end{array}\right]=\left[\begin{array}{lll}
K+\frac{4}{3} G & K-\frac{2}{3} G & K-\frac{2}{3} G \\
K-\frac{2}{3} G & K+\frac{4}{3} G & K-\frac{2}{3} G \\
K-\frac{2}{3} G & K-\frac{2}{3} G & K+\frac{4}{3} G
\end{array}\right]\left[\begin{array}{l}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33}
\end{array}\right]-3 K \epsilon_{o}\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right]
$$

Considering the special case of plane stress, $\sigma_{33}=0$, the stress-strain relations reduce in the presence of initial volumetric strains to:

$$
\left[\begin{array}{l}
\sigma_{11}  \tag{4.32}\\
\sigma_{22}
\end{array}\right]=\frac{E}{1-\nu^{2}}\left[\begin{array}{ll}
1 & \nu \\
\nu & 1
\end{array}\right]\left[\begin{array}{l}
\epsilon_{11} \\
\epsilon_{22}
\end{array}\right]-\frac{E}{1-\nu} \epsilon_{o}\left[\begin{array}{l}
1 \\
1
\end{array}\right]
$$

where the shear components are not affected by the temperature change in the case of isotropy.

### 4.1.2 Free Thermal Expansion

Under stress free conditions the thermal expansion $\boldsymbol{\epsilon}_{o}=\alpha\left[T-T_{o}\right] \mathbf{1}$ leads to $\boldsymbol{\epsilon}=\boldsymbol{\epsilon}_{o}$, i.e.

$$
\begin{align*}
& \epsilon_{11}=\alpha\left[T-T_{o}\right]  \tag{4.33}\\
& \epsilon_{22}=\alpha\left[T-T_{o}\right]  \tag{4.34}\\
& \epsilon_{33}=\alpha\left[T-T_{o}\right] \tag{4.35}
\end{align*}
$$

Thus the change of temperature results in free thermal expansion, while the mechanical stress remains zero under zero confinement, $\boldsymbol{\sigma}=\mathcal{E}: \boldsymbol{\epsilon}_{e}=\mathbf{0}$.

### 4.1.3 Thermal Stress under Full Kinematic Restraint

In contrast to the unconfined situation above, the thermal expansion is equal and opposite to the elastic strain $\boldsymbol{\epsilon}_{e}=-\boldsymbol{\epsilon}_{o}$ under full confinement, when $\boldsymbol{\epsilon}=0$. In the case of plane stress, the temperature change $\Delta T=T-T_{o}$ leads to the thermal stresses

$$
\begin{align*}
\sigma_{11} & =-\frac{E}{1-\nu} \alpha\left[T-T_{o}\right]  \tag{4.36}\\
\sigma_{22} & =-\frac{E}{1-\nu} \alpha\left[T-T_{o}\right] \tag{4.37}
\end{align*}
$$

