Applied Spatial Statistics: Lecture 7 Spatial inference

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Outline

- Conditional estimate and Kriging
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Estimating a curve or surface.

The additive statistical model:

Given n pairs of observations (x_i, y_i), $i = 1, \ldots, n$

$$\boldsymbol{y} = \boldsymbol{f} + \boldsymbol{\epsilon}$$

$\boldsymbol{\epsilon}_i$'s are random errors $E[\boldsymbol{\epsilon}] = 0$ COV($\boldsymbol{\epsilon}$) $= \sigma^2 I$

and f is an unknown smooth function:

$$f(\boldsymbol{x}) = \sum_{j=1}^{p} \phi_j(\boldsymbol{x})\boldsymbol{d}_j + g(\boldsymbol{x})$$

- $\{\phi_j\}$ *known* basis functions based on spatial locations
e.g. polyonimials or topography. $\boldsymbol{d}$ fixed coefficients but not known..

- $g(\boldsymbol{x})$ a Gaussian process expected value of 0 and a covariance function $\rho k_\theta(\boldsymbol{x}_1, \boldsymbol{x}_2)$ form of k is *known* expect for some parameter values:
ρ and θ that are unknown.

Summary

- Identify a fixed part of the spatial model and the covariance.

- Estimate the parameters by maximum likelihood – possibly profiling to make computations easier

- Find conditional mean for $g(\boldsymbol{x}^*)$ using MLE for parameters

- Find prediction for fixed part of model e.g. $\sum_j \phi(\boldsymbol{x}^*)\hat{\boldsymbol{d}}_j$

- Add two together for spatial prediction at $\boldsymbol{x}^*$

- Use Kriging covariance as the uncertainty – includes uncertainty for $\hat{\boldsymbol{d}}$

Tools for fitting a covariance

Grid search over the scale/range (θ) optimizing over σ and ρ

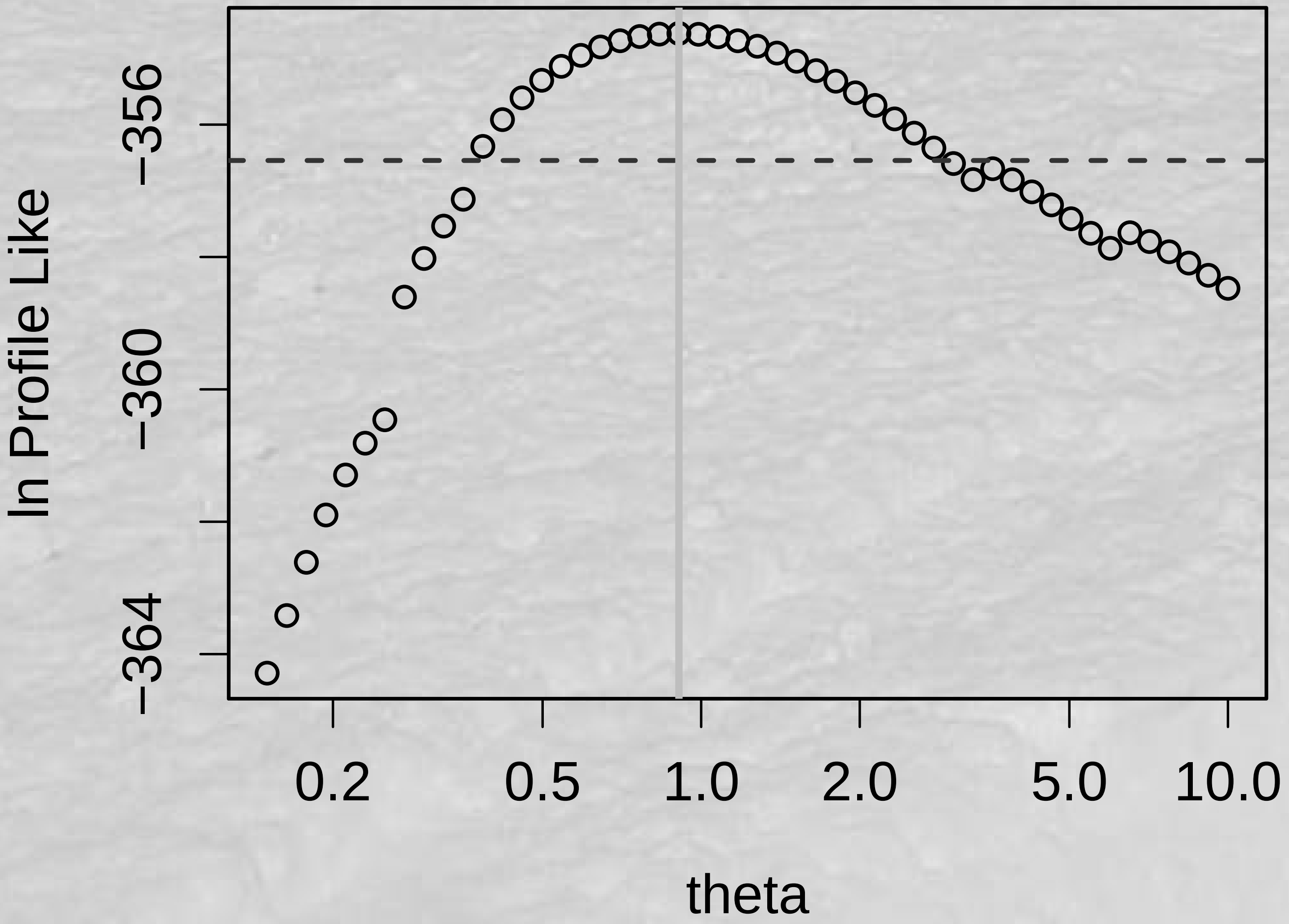
First try exponential covariance

```
theta.grid<- exp(seq(log(.15),log(10),,50))
MLE.fit0<-mKrig.MLE( xHW1, yHW1,Z= elevHW1,
                   Covariance="Matern",smoothness=.5,
                   par.grid=list( theta=theta.grid))

plot( theta.grid, MLE.fit0$summary[,2], log="x")

fit0<- mKrig(  xHW1, yHW1, Z= elevHW1,
                            Covariance ="Matern",smoothness=.5,
                            cov.args = MLE.fit0$cov.args.MLE,
                            lambda = MLE.fit0$lambda.best)
```

Profile over theta



Fitting over range and smoothness

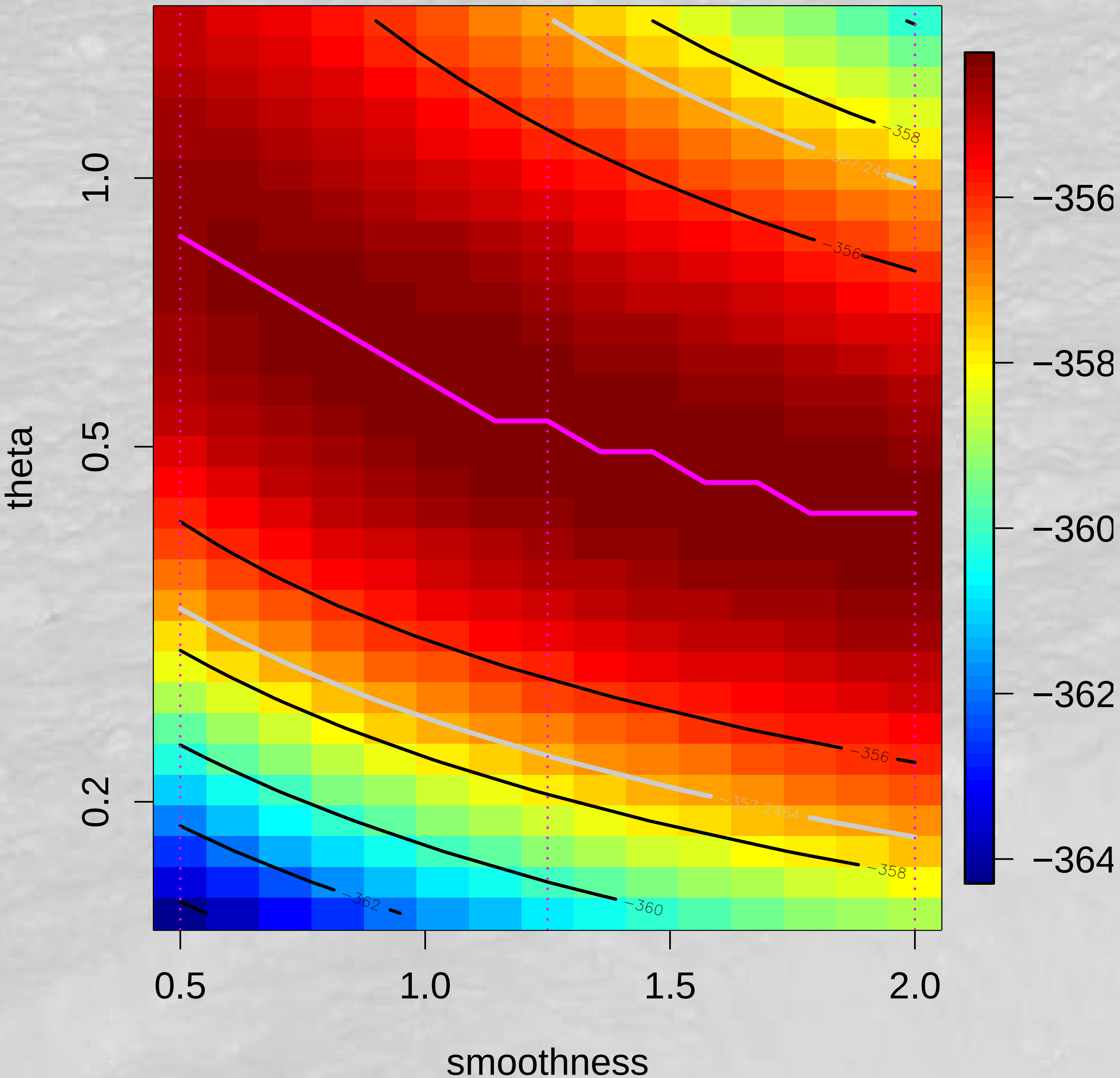
```
theta.grid<- exp(seq(log(.15),log(1.5),,15))
sm.grid<- seq( .5,2.0,,15)
par.grid<- make.surface.grid(
                          list( smoothness=sm.grid, theta=theta.grid))

MLE.fit1<-mKrig.MLE( xHW1, yHW1,Z= elevHW1,Covariance="Matern",
                                                par.grid=par.grid)

surface( as.surface( par.grid, MLE.fit1$summary[,2]), log="y" )
```

Profile likelihood surface

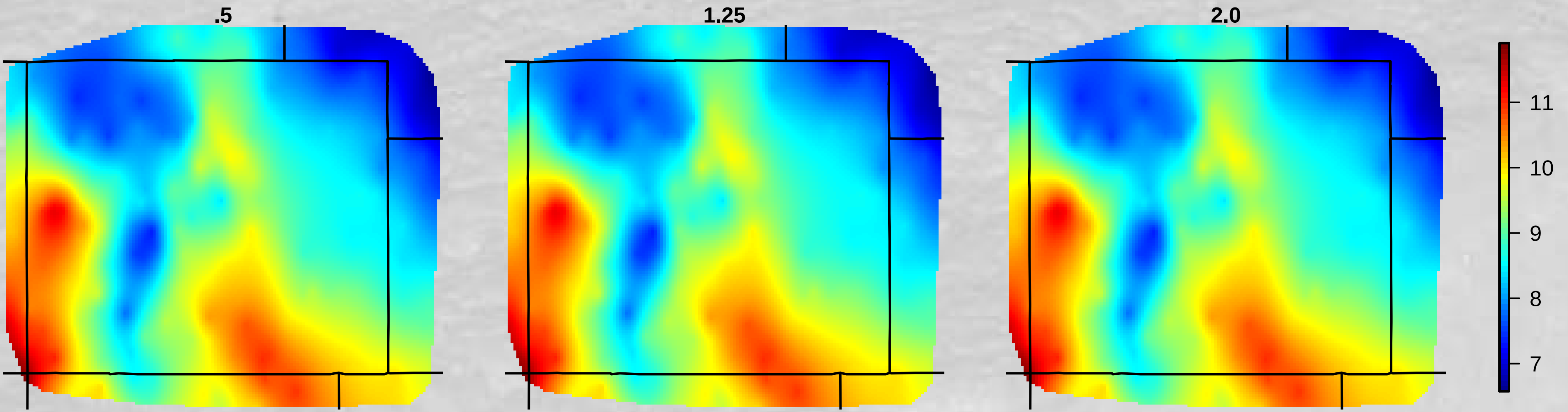
Marginal maxima, 95% confidence bounds



Comparing models

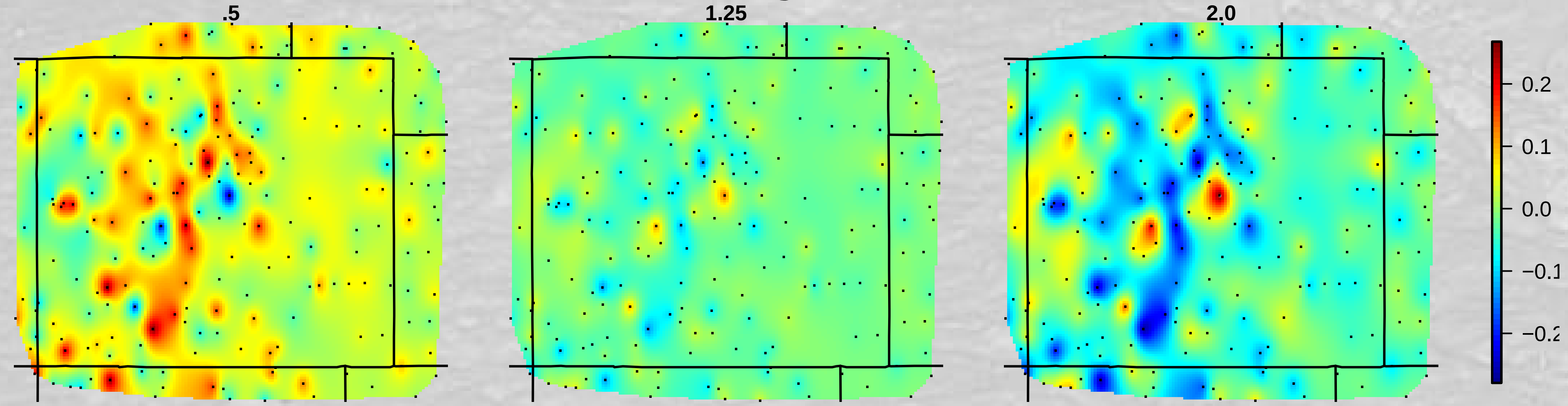
Look at smoothness .5, 1.25 and 2.0.

	OLS	.5	1.25	2.0
intercept	7.93723	10.23969	10.87367	11.02734
lon	-0.19823	-0.18444	-0.17551	-0.17267
lat	-0.49511	-0.53039	-0.52400	-0.52090
elev	-0.00596	-0.00568	-0.00565	-0.00564



Differences among models

Differences of fields from the average across the 3 cases:



Key parameter values (MLEs fixed smoothness)

```
          0.50    1.25    2.00
theta   0.8604  0.5343  0.4211
rho     0.9050  0.7485  0.7199
sigma   1.0258  1.1010  1.1188

eff.df 64.3066 42.0374 36.5950
```

ρ and θ adjust to give comparable estimates

Standard error of prediction

For fixed covariance parameters the spatial estimate is a linear combination of the observations:

- $\hat{\boldsymbol{d}} = \left(X^T M^{-1} X\right)^{-1} \left(X^T M^{-1}\right) \boldsymbol{y}$

  $M = (K + \lambda I)$

- $\hat{g}(\boldsymbol{x}^*) = \boldsymbol{w}^T \left(\boldsymbol{y} - X\hat{\boldsymbol{d}}\right)$

  $\boldsymbol{w} = \boldsymbol{k}^*(M)^{-1}$ and $\boldsymbol{k}_i^* = k(\boldsymbol{x}^*, \boldsymbol{x}_i)$

So combining all these expressions:

$$\hat{f}(\boldsymbol{x}^*) = \boldsymbol{\gamma}^T \boldsymbol{y}$$

Using linear statistical operations on random vectors one can verify that:

- $E(\hat{f}(\boldsymbol{x}^*) = \sum_j \phi(\boldsymbol{x}^*)\boldsymbol{d}_j$ Called an *unbaised estimate*

- $Z = f(\boldsymbol{x}^*) - \hat{f}(\boldsymbol{x}^*)$ the estimates error

Following Lecture 3:

$$VAR(Z) = \boldsymbol{\gamma}^T(\rho K + \sigma^2 I)\boldsymbol{\gamma} - 2\boldsymbol{\gamma}^T\boldsymbol{k}^* + VAR(g(\boldsymbol{x}^*))$$

- From Kriging results this is the smallest variance for different choices of γ

- If there were no fixed part of the model this is the conditional variance of g given $\boldsymbol{y}$

- Approximate Bayesian solution assuming known covariance (more on this later!)

A computational trick to find γ

To find $\boldsymbol{\gamma}_j$

- Create a synthetic set of observations where $y_i = 0$ for all i expect for $y_j = 1$

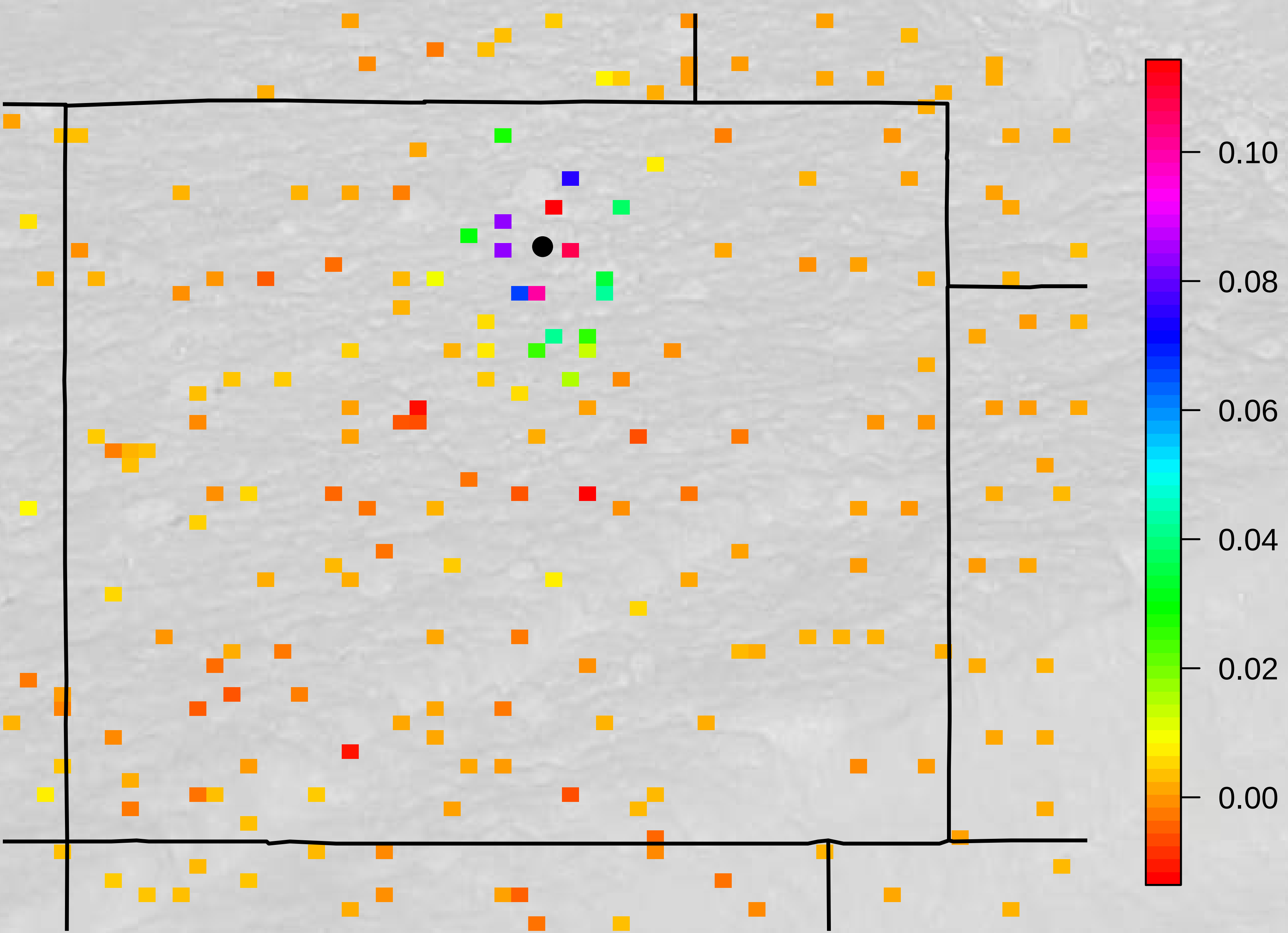
- Find the spatial prediction at $\boldsymbol{x}^*$: $\hat{f}(\boldsymbol{x}^*) = \boldsymbol{\gamma}^T \boldsymbol{y} = \gamma_j$

- repeat this computation to get all the components of γ

Automating in fields

```
fit2<- mKrig( xHW1, yHW1, Z= elevHW1,Covariance="Matern",
              smoothness=sm2, theta=theta2,
              lambda=exp(llambda2))
gamma.lyons<- predict( fit2, x.star,Z= elev.star, ynew= diag(1,nobs))
```

Visualizing weights for Lyons



More on Gamma weights

Can combine weights into rows of a matrix that maps observations into vector of predictions.

$$\begin{pmatrix} \hat{g}(\boldsymbol{x}_1) \\ \hat{g}(\boldsymbol{x}_2) \\ \vdots \\ \hat{g}(\boldsymbol{x}_n) \end{pmatrix} = \boldsymbol{\Gamma}\boldsymbol{y}$$

$\boldsymbol{\Gamma}$ called the smoothing matrix and is most sensitive to the parameter λ

Sum of diagonal elements of $\boldsymbol{\Gamma}$ is termed the effective degrees of freedom for the smoother.

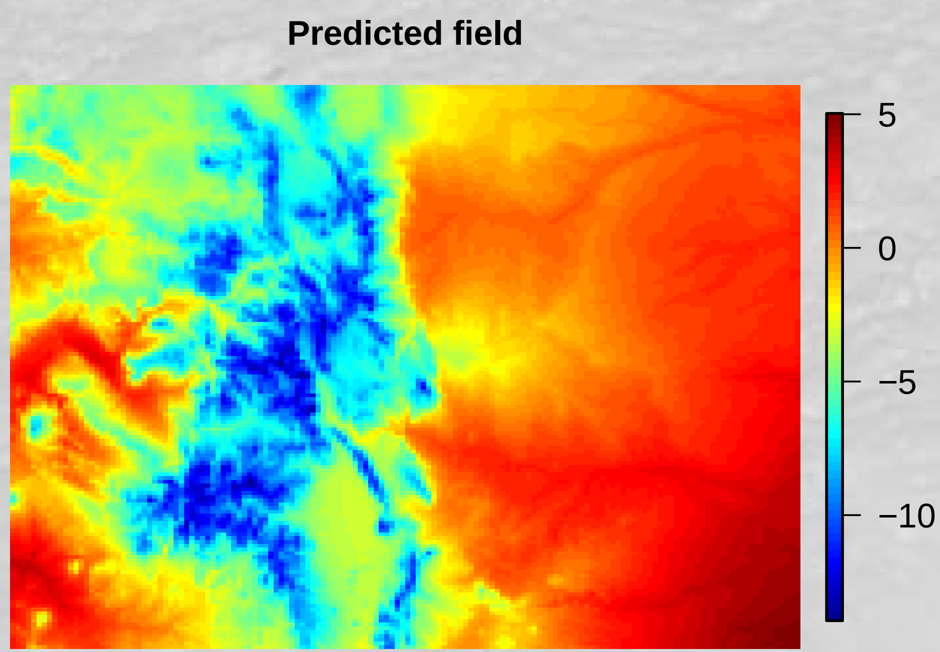
```
Gamma<- predict( fit2, xHW1, Z=elevHW1, ynew= diag( 1, nobs))
sum( diag( Gamma))
```

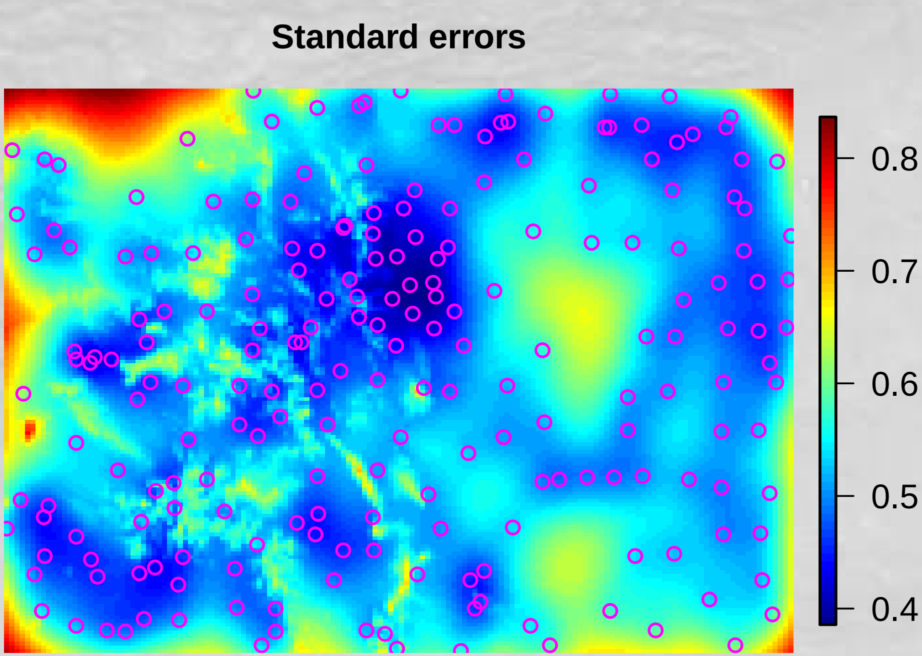
surface of prediction standard errors

```
glist<- fields.x.to.grid( xHW1, nx=150, ny=150)

xg<- make.surface.grid( glist)
data( RMelevation)
elevg<- interp.surface( RMelevation, xg)

sur2<- predict( fit2, xg, Z= elevg)
SE2<- predict.se( fit2, xg, Z= elevg)
```

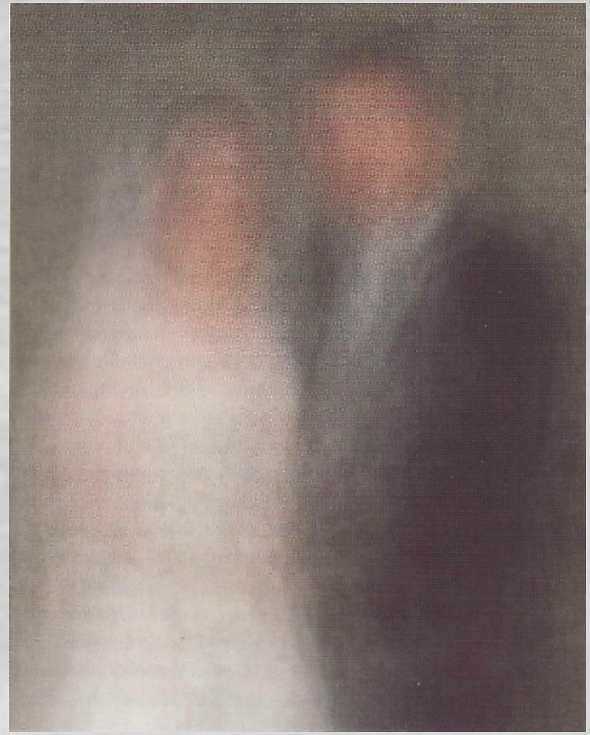




Inference beyond the mean

100 Weddings and the average

$\overline{wedding}$



J. Salavon *Cabinet* **15**